ENS - Master MVA / Paris 6 - Master Maths-Bio (2017-2018)

Tutorial 1

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Do not hesitate to contact me for more details concerning the solutions.

Exercice

Unicity

Using the Bernouilli principle, one can show that the water height of a tank with leaking hole at the bottom satisfies the following equation

$$\dot{h}(t) = -A\sqrt{h(t)}$$

where A is a positive constant depending on the physical parameters of the problem. Show that there are multiple solutions to this equation satisfying the condition $h(T) = 0$ for $T > 0$. Knowing the state of the tank at time t_0, can we find the tank state at any time?

> The function $h(t) = \frac{A^2}{4}(T-t)^2$ if $t \leq T$ and 0 otherwise is a solution. $h = 0$ is another solution, the non uniqueness comes from $x \to \sqrt{x}$ not being Lip at 0.

Exercice

Existence theorem for HH equations

We consider the HH equations

$$C\dot{V} = I - \bar{g}_K n^4(V - E_K) - \bar{g}_{Na} m^3 h(V - E_{Na}) - \bar{g}_L(V - E_L)$$

$$\dot{x} = \frac{1}{\tau_x(V)}(x_\infty(V) - x), \quad x \in \{m, n, h\}$$

Show that the solution is defined on $\mathbb{R}^+$.

> The vector field is bounded as $|\dot{V}| \leq A|V| + B$ where $A, B > 0$. This stems from the gating variables m, n, h belonging to $[0, 1]$. It is a well known result that the flow is defined on $\mathbb{R}^+$, *e.g.* by using Volterra result or Gronwall lemma (to show the non-explosion of the flow).

Exercice

The θ-neuron

It is an abstract model of spike generation. The potential is described by $\theta \in \mathcal{S}^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$ and satisfies:

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I$$

where I is the injected current. We consider that a spike is emitted when θ crosses the point $\theta = \pi$.

1. Show that for $I < 0$, there are two equilibria for the system, one stable and the other unstable. Show that every solution not starting at the unstable equilibrium converge to the stable equilibrium.
2. In the case $I > 0$, show that there is no equilibrium. Conclude that the trajectories are periodic orbits with regular spiking.
3. What happens when $I = 0$?

> 1) We find $\theta_{\pm} = \pm acos\frac{1+I}{1-I}$ hence producing two solutions $\theta_{\pm}$. Writing the equation as $\dot{\theta} = F(\theta)$, the (nonlinear) stability of the points is given by the sign of the real part of the eigenvalues of $dF(\theta_{\pm})$. One finds a single eigenvalue $\lambda_{\pm} = \pm\sqrt{-I}$. Hence θ_+ is unstable. In between those points, the vector field does not vanish, this is used to show the convergence to the fixed point θ_-. 2) similar to 1). 3) There is a saddle-node bifurcation, the unique equilibrium $\theta = 0$ is unstable.

Exercice

Non-uniqueness of the center manifold

Consider the system

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y. \end{cases}$$

Show that $\mathcal{M}(\beta) = \{(x, y) \mid y = A(x)\}$ with $A(x) = \beta e^{\frac{1}{x}}$ si $x < 0$ and $A(x) = 0$ otherwise, is a family of center manifolds.

(Hint) You may write the equation satisfied by the center manifold.

> Writing the equation satisfied by the the center manifold A, we find $x^2 A'(x) = -A(x)$. Solving this ODE gives the solution.

Exercice

Existence of invariant manifolds

We consider

$$(1)\ \ \dot{u} = F(u),$$

$F \in C^k(R^n, \mathbb{R}^n),\ k \geq 2$ with $F(0) = 0$. We write $L = dF(0)$ and decompose $\mathbb{R}^n = E_+ \oplus E_-$ such that $sign\Re\Sigma(L_\pm) = \pm$. Finally, we assume that $\{\lambda \mid |\Re\lambda| \leq \gamma\} = \emptyset$.

We want to find stable / unstable invariant manifolds. To this end, we define the space $\mathcal{E}^0 = C_b^0(\mathbb{R}_-, \mathbb{R}^n)$ and $\mathcal{E}^1 = C_b^1(\mathbb{R}_-, \mathbb{R}^n)$ for the case of the unstable manifold. We write

$$\mathcal{V}_u = \{Z(0) \in \mathcal{V}(0) \mid Z \in \mathcal{E}^0 and\ u\ sol.\ de\ (1)\}$$

We want to prove that $\mathcal{V}_u$ is an invariant manifold of class C^k which is tangent to E_+.

1. Show that $\|e^{L_+t}x\| \leq ke^{\gamma t}\|x\|$ for $t \leq 0$ and $x \in E_+$. Show that $\|e^{L_-t}x\| \leq ke^{-\gamma t}\|x\|$ for $t \geq 0$ and $x \in E_-$.

2. Define $\mathbb{A} : Z \in \mathcal{E}^1 \to \frac{dZ}{dt} - LZ \in \mathcal{E}^0$. Show that $\mathbb{A}$ is a continuous linear operator. Find its kernel. Show that it is sujective on $\mathcal{E}^0$.

3. We define $\tilde{\mathbb{A}}^{-1}f(t) = \int_0^t e^{L_+(t-s)}P_+f(s)ds + \int_{-\infty}^t e^{L_-(t-s)}P_-f(s)ds$ for $t \leq 0$. Show that it is continuous, $\tilde{\mathbb{A}}^{-1} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$.

4. Define Π on $\mathcal{E}^0$ by $\Pi Z(t) = e^{L_+t}P_+Z(0)$. Show that it is a projector on $\ker \mathbb{A}$. Show that $\Pi\tilde{\mathbb{A}}^{-1} = 0$.

5. Write $F(x) = Lx + R(x)$ with $R(x) = O(\|x\|^2)$. Show that a solution of (1) in $\mathcal{E}^1$ can be rewritten as a solution of $\tilde{\mathbb{A}}V = R(U + V)$. Solve this equation in a neighbourhood of 0. Conclude that $V = \Phi(U) = O(\|U\|^2)$.

6. Deduce from the previous question that $Z(0) = X + \Psi^+(X)$ for some $X \in E_+$ and $\Psi^+ \in C^k(E_+, E_-)$. **This is the equation of the unstable manifold $\mathcal{V}_u$.**

7. Show that $\mathcal{V}_u$ is invariant by the flow of (1).

8. Write the integral formulation in the case of $\mathcal{V}_s$.

> 1) $\ker \mathbb{A} = \{g \in \mathcal{E}^1,\ g(t) = e^{Lt}g_0,\ t \leq 0,\ g_0 \in E_+\}$. A solution of $\mathbb{A}Z = f \in \mathcal{E}^0$ is $Z = \tilde{\mathbb{A}}^{-1}f$. 5) Write $Z = U + V$ with $U = \Pi Z$, $V = (I - \Pi)Z$. Then $AZ = R(Z)$ reads as stated. We solve the equation in V using the implicit functions theorem. 6) Take $t = 0$ in V and use that $U = \Pi Z$. We find that $\Psi^+(X) = \Phi(e^{L_+t}X)(0)$. 7) This results from the fact that equation (1) is autonomous. Hence, the solutions commute with the translation in time. Indeed, write $t = t' + \tau$ and $X(t) = \tilde{Z}(t')$ in the equation $Z(t) = e^{L_+t}X + \tilde{\mathbb{A}}^{-1}R(Z(t))$ and show that $\tilde{Z}(0)$ is as in 6). 7) In this case, the problem reads

$$Z(t) = e^{L_- t} Y + \int_0^t e^{L_-(t-s)} P_- R(Z(s)) ds - \int_t^\infty e^{L+-(t-s)} P_+ R(Z(s)) ds$$

for $t \geq 0$.