ENS - Master MVA / Paris 6 - Master Maths-Bio (2017-2018)

Tutorial 2

Romain VELTZ, romain.veltz@inria.fr

Exercice

Wilson-Cowan

This is the main mechanism to produce cortical oscillations with two interacting populations (PING mechanism). Consider two populations E/I with the following dynamics

$$\begin{cases} \dot{E} & = & -E + S(J_{EE}E + J_{EI}I + \theta_E) \\ \dot{I} & = & -I + S(J_{IE}E + J_{II}I + \theta_I) \end{cases}$$

where S is the sigmoid function

$$S(x) = \frac{1}{1+e^{-x}}.$$

1. Write the equation for the equilibrium
2. Write the jacobian of the system (Hint: $S' = S(1-S)$)
3. Write the **linear** conditions for the Hopf bifurcation and find a way to compute the Hopf bifurcation curves in the plane (θ_E, θ_I).
4. Can we do the same for the Saddle-Node bifurcation curve?

Exercice

Delayed inhibition

This is the main mechanism to produce cortical oscillations with inhibitory neurons (ING mechanism). Consider one population of such neurones I with the following dynamics

$$\tau \dot{I}(t) \quad = \quad -I(t) + JS(\sigma I(t-D) + \theta)$$

where S is the sigmoid function (see above), σ is the nonlinear gain and $J < 0$.
1. Show that there is a unique negative stationary state I_σ^{stat} that is monotonic in σ.
2. Write the linear equation around I_σ^{stat} and look for perturbation $e^{\lambda t}U$. Find an equation for λ.
3. Write the solutions of this equations using the different solutions $W_k(z)$ of the equation $we^w = z$. This function is called the Lambert function. You have computed the spectrum.
4. We change of method. Give a necessary condition on $\sigma JS'$ in order to have a Hopf bifurcation. In this case, show that the critical delay is $D = \frac{1}{\sqrt{J^2-1}}\left(\pi - \arccos\left(\frac{1}{|J|}\right)\right)$.
5. Show that $\sigma \to \sigma S'(\sigma I_\sigma^{stat} + \theta)$ is increasing. Conclude on the existence of a Hopf bifurcation when increasing the nonlinear gain.

Exercise

Normal and center manifold

We consider $\frac{du}{dt} = \mathbf{A}u + \mathbf{R}(u,\mu)$. Let us write the Taylor expansion of $\mathbf{R}$ for a given p:

$$\mathbf{R}(u) = \sum_{2\le q+l\le p} \mathbf{R}_{ql}[u^{(q)}, \mu^{(l)}] + o\left(\|u\|^p\right),\ \mathbf{R}_{01} = 0$$

with $\mathbf{R}_{ql} = \frac{1}{q!l!}\frac{\partial \mathbf{R}}{\partial^q u \partial^l \mu}$, $u^{(q)} \equiv (u, \cdots, u) \in \mathcal{Z}_h^q$ and $\mu^{(l)} \equiv (\mu, \cdots, \mu) \in (\mathbb{R}^{m_{par}})^l$.

1. Assume that there is a center manifold Ψ and that we perform a Normal form simplification on the center manifold. Write the equation satisfied by the combined change of variables.
2. We consider the ring model $\dot{V} = -V + J \star S_0(V)$ on the circle. Assume that there a Pitchfork bifurcation at $\mu = \mu_0$. Compute the normal form as function of the parameters of the model.

> 1. we build a reduced equation for $u_c \in \mathcal{X}_c$ with the center manifold correction Ψ:
>
> $$u = u_c + \Psi(u_c, \mu), \quad \Psi(u_c,\mu) \in \mathcal{Z}_h.$$
>
> This reduced equation is
>
> $$\frac{du_c}{dt} = \mathbf{A}u_c + P_c\mathbf{R}\left(u_c + \Psi(u_c,\mu), \mu\right).$$
>
> Then, we apply a change of variable to u_c
>
> $$u_c = v_0 + \Phi_\mu(v_0), \quad v_0 \in \mathcal{X}_c$$
>
> to bring the reduced equation to a normal form given by:
>
> $$\frac{dv_0}{dt} = \mathbf{A}|_{\mathcal{X}_c} v_0 + \mathbf{N}_\mu(v_0) + \rho(v_0,\mu),$$
>
> where $\mathbf{N}_\mu$ is a polynomial of some degree p such that $\mathbf{N}_0(0) = 0,\ D_v\mathbf{N}_0(0) = 0$ and $\rho(v_0,\mu) = o(\|v_0\|^p)$. We write

$$u = v_0 + \tilde{\Psi}(v_0, \mu), \quad \tilde{\Psi}(v_0, \mu) \equiv \Phi_\mu(v_0) + \Psi(v_0 + \Phi_\mu(v_0), \mu) \in \mathcal{Z}$$

The nonlinear function $\tilde{\Psi}$ is solution of the next equations:

$$(NF): \begin{cases} D_{v_0}\tilde{\Psi}(v_0, \mu)\mathbf{A}|_{\mathcal{X}_c} v_0 - \mathbf{A}\tilde{\Psi}(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}(v_0) \\ \mathbf{Q}(v_0) \equiv \Pi_p \left[\mathbf{R}(v_0 + \tilde{\Psi}(v_0, \mu), \mu) - D_{v_0}\tilde{\Psi}(v_0, \mu)\mathbf{N}_\mu(v_0)\right] \end{cases}$$

where Π_p is the operator which takes the first $p+1$ terms in the Taylor expansion in the variable v_0.

2. By symmetry arguments, the bifurcation is of "pitchfork" type and the central part is given by $E_0 = Vect(\cos(2\cdot), \sin(2\cdot)) = \ker \mathbf{L}_{\mu_0}$ where $\mathbf{L}_\mu = -Id + \mu S'(\mu v_0^f(\mu))J$. We use complex coordinates $U_0 = A\zeta + c.c$ with $\zeta = e^{2i\theta}$ and $A \in C$, the normal form is given by:

$$\dot{A} = A\left(\frac{\sigma - \sigma_0}{\sigma_0} + \chi_3|A|^2\right) + h.o.t.$$

We write $V(\theta, t) = v_0^f + \mathbf{U}_0 + \Psi(\mathbf{U}_0, \mu)$. The Taylor expansion reads

$$\Psi(\mathbf{U}_0, \mu) = \Psi_{20}A^2 + \bar{\Psi}_{20}\bar{A}^2 + \Psi_{11}A\bar{A} + \Psi_{30}A^3 + \bar{\Psi}_{30}\bar{A}^3 + \Psi_{21}A^2\bar{A} + \bar{\Psi}_{21}A\bar{A}^2$$

We write $\mathbf{R}(\mathbf{U}, \mu) = -v_0^f - \mathbf{U} + \mathbf{J} \star S(\mu\mathbf{U} + \mu v_0^f) - \mathbf{L}_{\mu_0}$. We note that $D_{v_0}\tilde{\Psi}(v_0, \mu)\mathbf{A}|_{\mathcal{X}_c} v_0 = 0$. We look at the coefficient $A|A|^2$ in (NF), it satisfies

$$b\zeta - \mathbf{L}_{\mu_0}\Psi_{21} = 2\mathbf{R}_{20}(\zeta, \Psi_{11}) + 2\mathbf{R}_{20}(\bar{\zeta}, \Psi_{20}) + 3\mathbf{R}_{30}(\bar{\zeta}, \zeta, \zeta).$$

The coefficient A^2 gives

$$-\mathbf{L}_{\mu_0}\Psi_{20} = \mathbf{R}_{20}(\zeta, \zeta)$$

The coefficient $A\bar{A}$ gives

$$-\mathbf{L}_{\mu_0}\Psi_{11} = \mathbf{R}_{20}(\zeta, \bar{\zeta})$$

The coefficient A^3 gives

$$-\mathbf{L}_{\mu_0}\Psi_{30} = \mathbf{R}_{20}(\zeta, \Psi_{20}) + \mathbf{R}_3(\bar{\zeta}, \zeta, \zeta).$$

It remains to solve the above convolution equations. We write $e_n(\theta) = e^{2in\theta}$ and $J \star e_n = J_n e_n$. We find $\mathbf{R}_2(\zeta, \zeta) = \frac{1}{2}\mu_0^2 S^{(2)}(\mu_0 v_0^f)J \star \zeta^2$ hence
$\Psi_{20} = \alpha_{20}\zeta + \beta_{20}\bar{\zeta} + \frac{\mu_0^2 S^{(2)}(\mu_0 v_0^f)J_2}{2(-1+\mu_0 S^{(1)}(\mu_0 v_0^f)J_2} = \alpha_{20}\zeta + \beta_{20}\bar{\zeta} + \frac{\mu_0^2 S^{(2)}(\mu_0 v_0^f)J_2}{2(1-J_2/J_1)}$. The final result is

$$b = \mu_0^3 J_1 \left[\frac{S^{(3)}(\mu_0 v_0^f)}{3} + \mu_0 S^{(2)}(\mu_0 v_0^f)\left(\frac{J_0}{1 - J_0/J_1} + \frac{J_2}{2(1 - J_2/J_1)}\right)\right]$$