ENS - Master MVA / Paris 6 - Master Maths-Bio

Tutorial 2

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Exercice 1

Non-uniqueness of the center manifold

Consider the system

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y. \end{cases}$$

Show that $\mathcal{M}(\beta) = \{(x, y) \mid y = A(x)\}$ with $A(x) = \beta e^{\frac{1}{x}}$ si $x < 0$ and $A(x) = 0$ otherwise, is a family of center manifolds.

(Hint) You may write the equation satisfied by the center manifold.

Exercice 2

Analytic center manifold

Show that the following system

$$\dot{x} = -x^3$$
$$\dot{y} = -y + x^2$$

does not have an analytic centre manifold.

Exercice 3

Slow-Fast systems

Consider the general system

$$(E_\epsilon) : \begin{cases} \dot{x} = & f(x, y, \epsilon) \\ \dot{y} = & \epsilon g(x, y, \epsilon) \end{cases} \qquad \left(\text{recall Fitzhugh-Nagumo} \begin{cases} \dot{v} = & v - \frac{v^3}{3} - w + I \\ \dot{w} = & \epsilon\,(v + a - bw) \end{cases}\right)$$

The *critical manifold* is defined by $\mathcal{S} = \{(x, y) \mid f(x, y, 0) = 0\}$. It corresponds to a set of equilibria for the *layer* problem (E_0).

1. Give a sufficient condition for the existence of a continuous function $h : \mathcal{D}_0 \to \mathbb{R}^m$ with $\mathcal{D}_0$ connected having a non-empty interior such that $\mathcal{M} \equiv \{(h(y), y),\ y \in \mathcal{D}_0\} \subset \mathcal{S}$. $\mathcal{M}$ is called a *slow manifold*.
2. We assume that $\mathcal{M}$ is uniformly hyperbolic. That is: $\exists \sigma > 0$ such that for all $y \in \mathcal{D}_0$, the eigenvalues are bounded away from zero: $\max\limits_{\lambda \in \Sigma(\partial_x f(h(y), y))} |\Re \lambda| > \sigma$. Prove that there exists for ϵ small enough, a locally invariant manifold close to $(h(y_0), y_0)$ for some $y_0 \in \mathcal{D}_0$:

$$\mathcal{M}_\epsilon = \{(x, y) : x = h(y, \epsilon),\ y \in \mathcal{V}(y_0)\}$$

   where $h(y, \epsilon) = h(y) + \mathcal{O}(\epsilon)$.
3. (difficult) Show that $\mathcal{M}_\epsilon$ is uniformly asymptotically stable if $\mathcal{M}$ is.

Exercice 4

Adaptive exponential integrate-and-fire model (AdExp)

We consider the model

$$(1) \begin{cases} C\dot{V} = -g_L(V - E_L) + g_L \Delta_T \exp\left(\frac{V - V_T}{\Delta_T}\right) - w + I = F(V) - w + I \\ \tau_w \dot{w} = a(V - E_L) - w. \end{cases}$$

When the membrane potential V is high enough, the trajectory quickly diverges because of the exponential term. When a spike occurs, the membrane potential is instantaneously reset to some value Vr and the adaptation current is increased:

$$\begin{cases} V \to V_r \\ w \to w + b. \end{cases}$$

For simplicity, we restrict to the case $C = b = 1$.

1. Draw the nullclines. Discuss the dynamics according to the initial condition.
2. Write the equations satisfied by the equilibria.
3. Write $G_b(v) = F(v) - a(V - E_L)$. Show that G_b is strictly convex with a unique minimum $m(\mu)$ that is attained for $V = V^*(\mu)$

4. Show that m, v^* are at least C^2.
5. Compute the number of equilibria and their stability as function of I and b. In particular, show that:

    1. For $I > -m(b)$, there are no equilibria
    2. For $I = -m(b)$, there is a unique equilibrium $(v^*(b), w^*(b))$ which is not hyperbolic. It is unstable if $b > a$.
    3. If $I < -m(b)$, there are two equilibria $\left(v_-(I,b), v_+(I,b)\right)$ such that

    $$v_-(I,b) < v^*(b) < v_+(I,b)$$

    The equilibrium $v_+(I,b)$ is a saddle fixed point and the stability of $v_-(I,b)$ depends on I and $sign(b-a)$. If $b < a$ then $v_-(I,b)$ is attracting. If $b > a$ there is a smooth curve $I^*(a,b)$ defined implicitly by $F'\left(v_-(I^*(a,b)), b\right) = a$ such that if $I < I^*(a,b)$ then the equilibrium is attracting and if $I > I^*(a,b)$ then the equilibrium is repulsive.

6. Show that the curve $\{(b,I) \mid I = -m(b), a \neq b\}$ is a saddle-node bifurcation curve (when $F''(v^*) \neq 0$). Give the equation on the center manifold.
7. Assume $a > b$ and write v_a the unique solution of $F'(v_a) = a$. If $F''(v_a) \neq 0$, show that there is a Andronov-Hopf bifurcation at v_a on the curve $AH = \{(b,I); b > a \text{ and } I = bv_a - F(v_a)\}$.
8. Take $a > 0, b = a$ and assume that $F'''(v_a) \neq 0$. Show that the system has a Bogdanov-Takens bifurcation.

Exercice 5

Normal Form

The idea is to find a polynomial CHV which improves locally a nonlinear system, in order to analyze its dynamics more easily. We consider a differential equation:

$$\dot{x} = \mathbf{L}x + \mathbf{R}(x;\alpha),\ \mathbf{L} \in \mathcal{L}(\mathbb{R}^n),\ \mathbf{R} \in C^k(\mathcal{V}_x \times \mathcal{V}_\alpha, \mathbb{R}^m) \quad (1)$$

$$\mathbf{R}(0;0) = 0,\ d\mathbf{R}(0;0) = 0$$

The normal form theorem is the following:

> Then, $\forall p \in [2,k]$, there are neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbf{L}^n$ and $\mathbf{L}^m$, respectively, such that for any $\alpha \in \mathcal{V}_2$, there is a polynomial $\Phi_\alpha : \mathbf{L}^n \to \mathbf{L}^n$ of degree p with the following properties:
>
> 1. The coefficients of the monomials of degree q in Φ_α are functions of α of class C^{k-q}, and
>
> $$\Phi_0(0) = 0,\ d\Phi_0(0) = 0$$

2. For any $x \in \mathcal{V}_1$, the polynomial **Change of Variable** $x = y + \Phi_\alpha(y)$ transforms (1) into the normal form

$$\dot{y} = \mathbf{L}y + \mathbf{N}_\alpha(y) + \rho(y, \alpha)$$

where $\mathbf{N}_\alpha : \mathbf{L}^n \to \mathbf{L}^n$ is a polynomials of degree p
3. The coefficients of the monomials of degree q in $\mathbf{N}_\alpha$ are functions of α of class C^{k-q}, and

$$\mathbf{N}_0(0) = 0,\ d_x\mathbf{N}_0(0) = 0$$

4. the equality $\mathbf{N}_\alpha(e^{t\mathbf{L}^*}y) = e^{t\mathbf{L}^*}\mathbf{N}_\alpha(y)$ holds for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in \mathcal{V}_2$
5. the maps ρ belongs to $C^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and $\forall \alpha \in \mathcal{V}_2,\ \rho(y; \alpha) = o(|y|^p)$

Consider the case Hopf case: $\mathbf{L} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, $\omega > 0$. In the basis $(\zeta, \bar{\zeta}), \zeta = (1, -i)$: $\mathbf{L} = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$.

Write $x = y + \Phi_\alpha(y)$, the change of variable with $y = A\zeta + \overline{A\zeta}$

1. Prove that $\mathbf{N}_\alpha(A\zeta + \overline{A\zeta}) = AQ_\alpha(|A|^2)\zeta + \overline{AQ_\alpha}(|A|^2)\bar{\zeta}$ where Q_α is a polynomials.
2. Write the vector field at order 3 in A . Do you recognize something?