ENS - Master MVA / Paris 6 - Master Maths-Bio (2018-2019)

Tutorial 5

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Exercice

Wilson-Cowan

This is the main mechanism to produce cortical oscillations with two interacting populations (PING mechanism). Consider two populations E/I with the following dynamics

$$\begin{cases} \dot{E} & = & -E + S(J_{EE}E + J_{EI}I + \theta_E) \\ \dot{I} & = & -I + S(J_{IE}E + J_{II}I + \theta_I) \end{cases}$$

where S is the sigmoid function

$$S(x) = \frac{1}{1 + e^{-x}}.$$

1. Write the equation for the equilibrium
2. Write the jacobian of the system (Hint: $S' = S(1 - S)$)
3. Write the **linear** conditions for the Hopf bifurcation and find a way to compute the Hopf bifurcation curves in the plane (θ_E, θ_I).
4. Can we do the same for the Saddle-Node bifurcation curve?

Exercice

Delayed inhibition

This is the main mechanism to produce cortical oscillations with inhibitory neurons (ING mechanism). Consider one population of such neurones I with the following dynamics

$$\tau\dot{I}(t) \quad = \quad -I(t) + JS(\sigma I(t-D) + \theta)$$

where S is the sigmoid function (see above), σ is the nonlinear gain and $J < 0$. 1. Show that there is a unique negative stationary state I_σ^{stat} that is monotonic in σ. 2. Write the linear equation around I_σ^{stat} and look for perturbation $e^{\lambda t}U$. Find an equation for λ. 3. Write the solutions of this equations using the different solutions $W_k(z)$ of the equation $we^w = z$. This function is called the Lambert function. You have computed the spectrum. 4. We change of method. Give a necessary condition on $\sigma JS'$ in order to have a Hopf bifurcation. In this case, show that the critical delay is $D = \frac{1}{\sqrt{J^2-1}}\left(\pi - \arccos\left(\frac{1}{|J|}\right)\right)$. 5. Show that $\sigma \to \sigma S'(\sigma I_\sigma^{stat} + \theta)$ is increasing. Conclude on the existence of a Hopf bifurcation when increasing the nonlinear gain.

Exercise

Introduction to Mean field [Giacomin-etal:2011]

We here consider a network of N spiking neurons where each neuron is modelled with a θ-model. Hence, we consider the following dynamics

$$\frac{d}{dt}\theta_k = 1 - \cos\theta_k + (1 + \cos\theta_k)(\eta_k + I(t)),\ 1 \le k \le N.$$

The neuron k spikes when $\theta_k = \pi$. The neurons k affects the dynamics of its neighbors by $I(t) = \frac{J}{N}\sum_{j=1}^{N}(1-\cos\theta_j)^n$ with n large $J > 0$. The η_k are independent variables drawn from the distribution $g(\eta)d\eta$.

> Recall that by the change of variable $V_i = \tan\frac{\theta}{2}$, we end up with the Quadratic Integrate and Fire (QIF) model $\dot{V}_i = V_i^2 + I_i$ with appropriate boundary conditions.

1. Show **formally** that the measure $\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{(\theta_i,\eta_i)}$ converges to μ, solution of the PDE

$$\partial_t\mu = -\partial_\theta\left((1-\cos\theta + (1+\cos\theta)(\eta + I(t))\mu\right)$$

   where $I(t) = J\int\int(1-\cos\theta)^n\mu(d\theta, d\eta, t)$.
2. Show that the equilibrium, also called a *stationary distribution* when $\mu \ge 0, \int d\mu = 1$ can be written as

$$\mu^\infty(\theta,\eta) = \frac{C(\eta)}{1-\cos\theta + (1+\cos\theta)(\eta + I)}.$$

Express C as a function of I and find a scalar equation satisfied by I.

3. Show that there is a unique stationary distribution, *i.e.* a unique I^∞.
4. Write the linear equation around μ^∞ as $\partial_t \nu = A \cdot \nu$. Study whether 0 belongs to the spectrum of A. Discuss the (linear) stability of μ^∞.
5. (Difficult) Find the spectrum of A, or more precisely find an equation satisfied by the elements in the spectrum. **Hint:** one must solve the equation $\lambda \nu - A \cdot \nu =$ and find the $\lambda \in \mathbb{C}$ for which this equation has no solution. Note that this equation is a simple ODE...