ENS - Master MVA / Paris 6 - Master Maths-Bio (2018-2019)

Tutorial 4

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Exercice

Around the normal form theorem

We study the **Bogdanov-Takens** Normal form. It is the normal form of a system where a Saddle-node bifurcation and a Hopf bifurcation co-exist. Assume that the jacobian of a DS at $u = 0$ is given by its Jordan normal form $L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ where $u = (A, B) \in \mathbb{R}^2$.

1. Show that the polynomial N of the normal form is [N(u)=(AP(A),BP(A)+Q(A))] where P, Q are real-valued polynomials, satisfying $P(0) = Q(0) = Q'(0) = 0$.
2. Using a change of variables, show that the DS can be modified into

$$\begin{cases} dA/dt = \mathcal{B} \\ d\mathcal{B}/dt = \mathcal{B}P_1(A) + Q_1(A) + \rho_1(A, \mathcal{B}) \end{cases}$$

Exercise

Around the neural field equation

We consider a NFE on a bounded domain $\Omega \subset \mathbb{R}^p$ with a sigmoid S nonlinearity:

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_{\mathbb{R}} w(x,y)S(V(y,t))dy.$$

1. We assume that $w \in C^0(\Omega^2, \mathbb{R})$. Prove existence / uniqueness of the solution in the space $\mathcal{C} = \mathcal{C}(\Omega, \mathbb{R})$. *Hint:* show that it is globally Lipschitz.

2. Show that the NL is C^1.

We focus on the case $w(x) = w_0 + w_1 \cos$ in $\Omega = (-\pi, \pi)$

1. Write the equations satisfied by the equilibrium. Are they finite dimensional?
2. Write a simplified set of equations for the dynamics.
3. Consider a stationary state. Can you study its stability despite the fact that the equations are infinite dimensional? (Can you find a case where you can...?)

Exercise

Around the neural field equation of Amari type

We consider a neural field equation on the real line

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_{\mathbb{R}} w(x-y)S(V(y,t))dy + h$$

in the case where $S(v) = \mathbf{1}_{v>0}$ is the Heaviside function and $h \in \mathbb{R}$. w is a real **even** function called the **connectivity kernel**. We defined $W(x) = \int_0^x w$. We further define $R(V) = \{x,\ V(x) > 0\}$. An equilibrium V^{eq} is said **localized** if $R(V^{eq}) = (a_1, a_2)$ with $a_i \in \mathbb{R}$. In this case, we can always assume $a_1 = 0$ by translation invariance.

1. An equilibrium V^{eq} such that $R(V^{eq}) = \emptyset$ exists if and only if $h < 0$.
2. An equilibrium V^{eq} such that $R(V^{eq}) = \mathbb{R}$ exists if and only if $2W_\infty > -h$ where $W_\infty := \lim_{x\to\infty} W(x)$.
3. An equilibrium V^{eq} such that $R(V^{eq}) = (0, a)$ exists if and only if $h < 0$ and $a > 0$ satisfies $W(a) + h = 0$.
4. Find solutions with a periodic support, namely $R(V^{weq}) = \cup_{n=-\infty}^{\infty}[-b + nL, b + nL]$ under the restriction $2b < L$. Find an equation satisfied by b.
5. Construction of traveling fronts $V(x,t) = U(x - ct)$ where the speed c and the waveform U have to be determined. One can introduce traveling wave coordinate $\xi = x - ct$ and assume $c > 0$.
6. **Interface dynamics**. We assume that a solution is such that $R(V(x,t)) = (-a(t), a(t))$ and that $0 \le V_0(x) \le 1$. We assume that V_0 is even. Find an equation satisfied by a in the case $h = 0$.
7. In the case of a non-convolutional kernel, $w(x,y) = e^{-|x-y|}(1 + a\cos(y))$, find the number of stationary solutions as function of their width.

> This behavior is called snaking of stationary solutions.