ENS - Master MVA / Paris 6 - Master Maths-Bio (2018-2019)

Tutorial 4

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Exercice

Around the normal form theorem

We study the **Bogdanov-Takens** Normal form. It is the normal form of a system where a Saddle-node bifurcation and a Hopf bifurcation co-exist. Assume that the jacobian of a DS at $u = 0$ is given by its Jordan normal form $L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ where $u = (A, B) \in \mathbb{R}^2$.

1. Show that the polynomial N of the normal form is [N(u)=(AP(A),BP(A)+Q(A))] where P, Q are real-valued polynomials, satisfying $P(0) = Q(0) = Q'(0) = 0$.
2. Using a change of variables, show that the DS can be modified into

$$\begin{cases} dA/dt = \mathcal{B} \\ d\mathcal{B}/dt = \mathcal{B}P_1(A) + Q_1(A) + \rho_1(A, \mathcal{B}) \end{cases}$$

> 1) We set N(u) = (Φ1(A,B),Φ2(A,B)) where Φ1 and Φ2 are polynomials in (A,B). Then we have $L^*(A, B) = (0, A)$ and , using the identity $DN(u)L^*u = L^*N(u)$ we obtain $A\partial_B\Phi_1 = 0$ and $A\partial_B\Phi_2 = \Phi_1$. Consequently, Φ1 does not depend upon B, Φ1(A,B) = ϕ1(A), and since the polynomial A∂ Φ2 /∂ B = Φ1 is divisible by A, there exists a polynomial P such that Φ1(A,B) = AP(A). Then the equation for the polynomial Φ2 leads to Φ2 (A, B) = BP(A) + Q(A) with Q a polynomial. Finally, we find that P(0) = Q(0) = Q′(0) = 0, since N(0) = 0 and DN(0) = 0 2) The previous question gives $\dot{A} = B + AP(A) + \rho_0(A, B)$ (similarly for $\dot{B}$). Making the change of variables $\tilde{B} = B + AP(A) + \rho 0(A, B)$, By the implicit function theorem, this change of variables is invertible $B = \tilde{B} - AP(A) + \tilde{\rho_0}(A, \tilde{B})$ which leads to the system with $P_1(A) = P(A) + \frac{d}{dA}(APA(A))$ and $Q_1(A) = Q(A) - A \cdot P(A)^2$.

Exercise

Around the neural field equation

We consider a NFE on a bounded domain $\Omega \subset \mathbb{R}^p$ with a sigmoid S nonlinearity:

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_\Omega w(x,y)S(V(y,t))dy.$$

1. We assume that $w \in C^0(\Omega^2, \mathbb{R})$. Prove existence / uniqueness of the solution in the space $\mathcal{C} = \mathcal{C}(\Omega, \mathbb{R})$. *Hint:* show that it is globally Lipschitz.
2. Show that the NL is C^1.

We focus on the case $w(x) = w_0 + w_1 \cos$ in $\Omega = (-\pi, \pi)$

1. Write the equations satisfied by the equilibrium. Are they finite dimensional?
2. Write a simplified set of equations for the dynamics.
3. Consider a stationary state. Can you study its stability despite the fact that the equations are infinite dimensional? (Can you find a case where you can...?)

> 1/ This is a consequence of Cauchy Lip. theorem. We easily check that the RHS $F(V)$ is globally Lipschitz by noting that $S_m := \sup S'(x) < \infty >$. Hence $\|F(V_1) - F(V_2)\|_\infty \leq (1 + S_m\|w\|_\infty)\|V_1 - V_2\|_\infty$. It implies that the NFE has a solution globally defined in time. 2/ $V \to V$ is C^1 hence we focus on $G(V) = W \cdot S(V)$. From Taylor with integral reminder, we have $S(V+U) - S(V) - US'(V) = \frac{1}{2}\int_0^1 (1-t)^2 S^{(2)}(V+tU)U^2 dt$. This shows that $W \cdot S(V+U) - W \cdot S(V) - W \cdot US'(V) = \frac{1}{2}\int_0^1 (1-t)^2 WS^{(2)}(V+tU)U^2$. On can check that the linear operator $\mathbf{L} = U \to W \cdot US'(V)$ is continuous on $\mathcal{C}$ and is thus the candidate for the differential of G at V. It remains to check that the integral term is $o(U)$. The reminder is $\frac{1}{2}\int_0^1 (1-t)^2 W \cdot S^{(2)}(V+tU)U^2 dt$ and its norm is bounded by $\sup_x S^{(2)}(x)\|w\|_\infty \|U\|^2 = o(U)$. This shows that G is differentiable.

Exercise

Around the neural field equation of Amari type

We consider a neural field equation on the real line

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_\mathbb{R} w(x-y)S(V(y,t))dy + h$$

in the case where $S(v) = \mathbf{1}_{v>0}$ is the Heaviside function and $h \in \mathbb{R}$.

The **connectivity kernel** $w \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a real **even** function which is integrable on $\mathbb{R}$. We define $W(x) = \int_0^x w$ and $W_\infty := \lim\limits_{x\to\infty} W(x)$.

We further define $R(V) = \{x,\ V(x) > 0\}$. An equilibrium V^{eq} is said **localized** if $R(V^{eq}) = (a_1, a_2)$ with $a_i \in \mathbb{R}$. In this case, we can always assume $a_1 = 0$ by translation invariance.

1. An equilibrium V^{eq} such that $R(V^{eq}) = \emptyset$ exists if and only if $h \leq 0$.
2. An equilibrium V^{eq} such that $R(V^{eq}) = \mathbb{R}$ exists if and only if $2W_\infty > -h$.
3. We here assume the following behavior of W on $\mathbb{R}^+$. W is strictly increasing towards its maximal value W_M and is then stricly decreasing and converging to $W_\infty < 0$. An equilibrium V^{eq} such that $R(V^{eq}) = (0, a)$ exists if and only if $h < 0$ and $a > 0$ satisfies $W(a) + h = 0$.
4. Find solutions with a periodic support, namely $R(V^{weq}) = \cup_{n=-\infty}^{\infty}[-b + nL, b + nL]$ under the restriction $2b < L$. Find an equation satisfied by b.
5. Construction of traveling fronts $V(x,t) = U(x - ct)$ where the speed c and the waveform U have to be determined. One can introduce traveling wave coordinate $\xi = x - ct$ and assume $c > 0$.
6. **Interface dynamics**. We assume that a solution is such that $R(V(x,t)) = (-a(t), a(t))$ and that $0 \leq V_0(x) \leq 1$. We assume that V_0 is even. Find an equation satisfied by a in the case $h = 0$.
7. In the case of a non-convolutional kernel, $w(x,y) = e^{-|x-y|}(1 + a\cos(y))$, find the number of stationary solutions as function of their width.

> This behavior is called snaking of stationary solutions.

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> 1/ If such solution exists then $V(x) = h$ which requires $h \leq 0$. On the contrary, if $h < 0$, then $V(x) = h$ is a stationary solution of the NFE.
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> 2/ If there is such solution then it satisfies $V(x) = \int_\mathbb{R} w(x-y)dy + h = 2W_\infty + h > 0$. On the contrary, if $2W_\infty + h > 0$, then $V(x) = 2W_\infty + h$ is such solution.
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> 3/ A localised solution is such that $V(x) = \int_0^a w(x-y)dy + h = W(x) - W(x-a) + h$. This solution is continuous, hence it satisfies $V(0) = V(a) = 0$ which implies $W(a) + h = 0$. Finally, $V \to h$ when $x \to \infty$ which implies $h \leq 0$. On the contrary, when $W(a) = h$ holds and $h \leq$, one finds $V(a) = V(0) = 0$. Moreover, such V is C^1. Using the hypothesis on the shape of W, one can prove that $V(x) = W(x) - W(x-a) + h$ is positive on the interval $(0, a)$ and negative elsewhere, provided that $h \leq 0$.
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> 4/ These solutions take the form $U_L(x) = \sum_{n\in\mathbb{Z}} \int_{-b+nL}^{b+nL} w(x-y)\mathrm{d}y + h = \sum_{n\in\mathbb{Z}}(W(x+b+nL) - W(x-b+nL)) + h$. The

threshold condition reads $U_L(\pm b + nL) = 0$. This gives $h + \sum_{n\in\mathbb{Z}}(W(2b+nL) - W(nL)) := h + \mathbf{W}_L(b) = 0$.

5/ the solution satisfies $U_f(\xi) = \mathrm{e}^{\xi/c}\left[\kappa - \frac{1}{c}\int_0^\xi \mathrm{e}^{-y/c}\,(W_\infty - W(y))\,\mathrm{d}y\right]$. Assuming $c > 0$ and requiring boundedness implies $\kappa = \frac{1}{c}\int_0^\infty \mathrm{e}^{-y/c}\,(W_\infty - W(y))\,\mathrm{d}y$. The TW is thus of the form $U_f(\xi) = \frac{1}{c}\int_0^\infty \mathrm{e}^{-y/c}\,(W_\infty - W(y+\xi))\,\mathrm{d}y$.

6/ By definition $V(\pm a(t), t) = 0$ hence $\pm x(t)a'(t) + \partial_t V(\pm a(t), t) = 0$ where we defined $\pm\alpha(t) = \partial_x u(\pm a(t), t)$. One gets $a'(t) = -\frac{1}{\alpha(t)}[W(2a(t)) - \kappa]$. This equation is not well defined for $\alpha(t) = 0$. We now get an expression for α. We define $z(x,t) = \partial_x V(x,t)$ and find $\partial_t z(x,t) = -z(x,t) + w(x + a(t)) - w(x - a(t))$ which allows to find $\alpha(t) = u_0'(a(t))\mathrm{e}^{-t} + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s[w(a(t)+a(s)) - w(a(t)-a(s))]\mathrm{d}s$. This gives us a closed system describing the evolution of a, α with the initial conditions $a(0) = l, \alpha(0) = u_0'(l)$ as long as $\alpha < 0$.