- Draft - Summary for the MMN lectures

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Abstract. This is a draft. Please do not distribute as many references are not cited properly.

1. Picard theorems

We start with a very standard result that is the basis for most results in these notes. This first section is based on [**?**].

1.1. Basic results

Definition 1.1

A function $T : X \to X$ where (X, d) is a metric space is contracting iff $\forall x, y \in X$ $d(Tx, Ty) \leq k \cdot d(x, y)$ where $k < 1$.

Theorem 1.1 (*Picard*)

Let X be a complete metric space and $T : X \to X$ a contracting mapping. Then T has a unique fixed point i.e. the unique solution of the equation $T(y) - y = 0$.

Proof. Take $x_0 \in X$ and show that $(T^n x_0)_n$ is a Cauchy sequence hence convergent (in a complete space), its limit will be the solution of the fixed point equation. If $n \geq m$

$$d(T^n x_0, T^m x_0) \leq d(T^m(T^{n-m}x_0), T^m x_0) \leq k^m d(T^{n-m}x_0, x_0).$$

We have from the triangle inequality:

$$d(T^{n-m}x_0, x_0) \leq \sum_{i=0}^{n-m-1} d(T^{i+1}x_0, T^i x_0) \leq \sum_{i=0}^{n-m-1} k^i d(Tx_0, x_0)$$

$$\leq \frac{1-k^{n-m}}{1-k} d(Tx_0, x_0) \leq \frac{1}{1-k} d(Tx_0, x_0).$$

This gives:

$$d(T^m x_0, T^n x_0) \leq \frac{k^m}{1-k} d(Tx_0, x_0)$$

which shows that $(T^n x_0)_n$ is a Cauchy sequence. ∎

We provide without proof a version with parameters. To have a notion of differentiability, we need to assume that we work in a Banach space. Hence, if $p = 0$, X is a metric space and a closed subspace of a Banach space if $p > 0$. Also, if $p = 0$, Λ is a metric space and an *open* subspace of a Banach space if $p > 0$. We consider a mapping $T \in C^p(X \times \Lambda, X)$ such that $x \to T(x, \lambda)$ is Lipschitz with constant $k(\lambda)$ where $\lambda \to k(\lambda)$ is continuous. We say that we have a C^p **map of contracting functions** if moreover $k(\lambda) < 1$.

Theorem 1.2 (*Picard with parameters*)

A C^p map of contracting functions on a space X has a family of fixed points $x(\lambda)$ where the map $\lambda \to x(\lambda)$ is C^p.

1.2. Applications, see [?]

We work here in Banach spaces, *i.e.* complete normed vector space. Recall that a linear map $L \in \mathcal{L}(E, F)$ between Banach spaces is continuous iff $\|L\|_{\mathcal{L}(E,F)} \equiv \sup_{\|x\|_E \leq 1} \|Lx\|_F < \infty$.

Theorem 1.3 (*Implicit functions theorem*)

*Let us consider two open sets U, V in Banach spaces E_1, E_2 and $f : U \times V \to F$ a C^k application with $k \geq 1$. We assume $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \in \mathcal{L}(E_2, F)$ has a **continuous** inverse. Then there are neighbourhoods U' of x_0, V' of y_0 and a mapping $\phi \in C^k(U', V')$ such that*

$$\forall (x, y) \in U' \times V', \ f(x, y) = 0 \Leftrightarrow y = \phi(x).$$

Proof. For‡ $x \in B(x_0, r)$ and $y \in B(y_0, r')$, the mapping

$$T_x(y) = y - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} f(x, y)$$

is continuous from $\overline{B(y_0, r')}$ to itself if r, r' are small enough. Indeed:

$$dT_x(y) = Id - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} \frac{\partial f}{\partial y} f(x, y)$$

is close to zero hence we can chose r, r' small enough so that for all x in $B(x_0, r)$ and y in $B(y_0, r')$, we have $\|dT_x(y)\| \leq \frac{1}{2}$. Then it gives $\|T_x(y_0) - y_0\| = \| \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} f(x, y)\| \leq r'/2$ if r, r' are small enough. It follows that $T_x\left(\overline{B(y_0, r')}\right) \subset \overline{B(T_x(y_0), r'/2)} \subset \overline{B(y_0, r')}$ (strict inclusion). Hence, $T_x : \overline{B(y_0, r')} \to \overline{B(y_0, r')}$ is contracting and we conclude with the Picard theorem with parameters. ■

We also have a result to invert a nonlinear map.

‡ We use the definition $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$

Theorem 1.4 (*Inverse function theorem*)

Let $\phi \in C^k(U,V)$ and $d\phi(x_0)$ has a bounded inverse. There are open sets U', V' containing x_0 and $y_0 = \phi(x_0)$ and a map $\psi \in C^k(V',U')$ such that $\phi \circ \psi = Id$.

Proof. We apply the previous theorem to $f(u,v) = v - \phi(v)$. ■

1.3. Cauchy-Lipschitz theorems

We consider $F : I \times \Omega \to E$ where Ω is an open set of a Banach space E and I is an open interval of $\mathbb{R}$. We want to solve

$$\dot{x} = F(t,x) \tag{1}$$

with the initial condition $x(t_0) = x_0 \in \Omega, t_0 \in I$.

Theorem 1.5 (*Cauchy-Lipschitz*)

We assume that F is continuous and Lipschitz in the second variable. Then for all $\tau \in I$ and $u_0 \in \Omega$ there exist $\delta, \alpha > 0$ such that the system

$$\begin{cases} \dot{x} = F(t,x), \\ x(t_0) = x_0 \in \Omega \end{cases}$$

has a unique solution defined on $]t_0-\alpha, t_0+\alpha[$ for all $x_0 \in B(u_0)$ and $t_0 \in]\tau-\delta, \tau+\delta[$.

Proof. Left in exercise, you may use the Picard theorem. ■

The theorem looks complicated in this form because we want that for t_0 in a neighbourhood of τ and x_0 in a neighborhood of u_0, the size 2α of the domain of definition is uniformly minored. We also give a version with parameters. Note that the initial condition can be taken as a parameter.

Theorem 1.6 (*Cauchy-Lipschitz with parameters*)

We assume that $F(\lambda,t,x)$ is in $C^k(\Lambda \times I \times \Omega, E)$ with $k \geq 0$, Lipschitz in the variable x and Λ satisfies the hypothesis of theorem 1.2. Let $x_\lambda(t;t_0,x_0)$ be the solution of

$$\begin{cases} \dot{x} = F(\lambda,t,x), \\ x(t_0) = x_0 \in \Omega \end{cases}$$

Then for all $(\lambda_0,\tau,u_0) \in \Lambda \times I \times \Omega$, there exist $\delta, \alpha > 0$ such that x_λ is defined on $]t_0-\alpha, t_0+\alpha[$ for all $(\lambda_0,t_0,x_0) \in B(\lambda_0,\delta) \times]\tau-\delta, \tau+\delta[\times B(u_0,\delta)$. Moreover, the map $(t,\lambda,t_0,x_0) \to x_\lambda(t;t_0,x_0)$ is C^k.

Up to the domain of definition, this theorem amounts to saying that $(t,\lambda,t_0,x_0) \to x_\lambda(t;t_0,x_0)$ is C^k.

2. Stability

We consider an autonomous dynamical system

$$x_{n+1} = F(x_n) \tag{2}$$

or

$$\begin{cases} \dot{x} = F(x), \\ x(t_0) = x_0 \in \Omega \end{cases}$$

and write ϕ the flow *i.e.* $x(t) = \phi^t(x_0)$ in the continuous case and $x_n = \phi^n(x_0)$ in the discrete case.

Definition 2.1

*An **equilibrium** is a point $x \in E$ such that $\phi^t(x) = x$ in the continuous or discrete case.*

In this section, we study the stability of an equilibrium.

Lemma 2.1 *([?])*

Let us consider $A \in \mathcal{L}(E)$ where E is a Banach space and there exists $b \geq 0$ such that all elements in the spectrum Σ of A satisfy $\sup\limits_{\lambda\in\Sigma} |\lambda| = b$. Then for all $\epsilon > 0$, we can choose a norm in E, equivalent to the given one, such that $\|A\|_{\mathcal{L}(E)} < (b + \epsilon)$.

Proof. Consider the new norm $\|x\|_{new}$ for $x \in \mathbb{R}^n$ given by $\|x\|_{new} = \sup\limits_{n\geq 0} \frac{\|A^n x\|}{(b+\epsilon)^n}$. It is a norm that satisfies $\|Ax\|_{new} \leq (b+\epsilon)\|x\|_{new}$ and which is equivalent to the given one. Indeed, we first have $\|x\| \leq \|x\|_{new}$ (for $n = 0$). Then we find $\|x\|_{new} \leq \sup\limits_{n\geq 0} \frac{\|A^n\|_{\mathcal{L}(E)}}{(b+\epsilon)^n} \|x\|$. The Gelfand spectral radius theorem [?][VII.3.4] gives $\lim\limits_n \sqrt[n]{\|A^n\|}_{\mathcal{L}(E)} = b$, there is a constant K such that $\|x\|_{new} \leq K \|x\|$ and the norms are equivalent. ∎

We can prove the previous lemma in finite dimensions using the Jordan decomposition (see exercises).

2.1. Case of equilibria

Definition 2.2

*An equilibrium x^{eq} is **stable** if given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\|x_0 - x^{eq}\| < \delta$, the solution of the initial value problem $x(t, 0, x_0)$ exists for all $t \geq 0$ and $\|x(t, 0, x_0) - x^{eq}\| < \epsilon,\ t \geq 0$.*

Definition 2.3

*An equilibrium x^{eq} is **asymptotically stable** if it is stable and there exists $b > 0$ such that if $\|x_0 - x^{eq}\| < b$, then $\lim_{t\to\infty} \|x(t, 0, x_0) - x^{eq}\| = 0$. It is **exponentially stable** iff $\exists V \in \mathcal{V}(x^{eq}), \gamma > 0, k \in (0, 1)$, $\forall x \in V$, $\|F^n(x) - x^{eq}\| \leq \gamma k^n$.*

We let the reader adapt these definitions to the case of discrete dynamical systems.

Theorem 2.1 (***Stability of equilibria for discrete DS, [?]***)

Let E be a Banach space and $F : E \to E$ be differentiable at an equilibrium x^{eq} and $dF(x^{eq}) \equiv A \in \mathcal{L}(E)$ is continuous. If the spectrum of A lies in a compact subset of the open unit disc, then x^{eq} is exponentially stable.

Proof. Using Lemma 2.1, we chose a norm, equivalent to the given one such that $\|A\|_{\mathcal{L}(E)} = b < 1$. We assume that $x^{eq} = 0$ for simplicity. Then, from the differentiability: $\forall\epsilon > 0, \exists\delta > 0$ such that $\|x\| \leq \delta$ gives $F(x) = Ax + R(x)$ with $\|R(x)\| \leq \epsilon \|x\|$. Hence $\|F(x)\| \leq (b+\epsilon)\|x\|$. We chose ϵ such that $b + \epsilon < 1$. Now $\|F^n(x)\| \leq (b+\epsilon)^n x \to 0$ and x^{eq} is exponentially stable. ∎

We could also have proved the theorem using the fact that $x \to dF(x)$ is continuous and the Picard theorem. We now consider an autonomous dynamical system $\dot{x} = F(x)$.

Theorem 2.2

Let $\mathcal{O} \subset E$ be an open set of a Banach space, and $F : \mathcal{O} \to E$ be C^1. Suppose that $x^{eq} \in \mathcal{O}$ is an equilibrium of F and that the spectrum Σ of $A \equiv dF(x^{eq}) \in \mathcal{L}(E)$ is such that $\sup\limits_{\lambda\in\Sigma} \Re\lambda = b < 0$. Then x^{eq} is asymptotically stable.

Proof. We assume that $x^{eq} = 0$. From theorem 1.6, $x \to \phi^t(x)$ is C^1. It is easy to see that the differential $d\phi^t(x_0)$ is equal to the solution in $\mathcal{L}(E)$ of $\dot{L} = A(t)L$ with $L(0) = Id$ and $A(t) = dF(\phi^t(x_0))$. We are interested in the case $x_0 = x^{eq}$ for which $A(t) = dF(x^{eq}) = A$. Hence, $d\phi^t(x^{eq}) = e^{At}$. The time-T step ϕ^T along orbits define a discrete (differentiable) dynamical system $x_{n+1} = \phi^T(x_n)$ with jacobian $e^{AT} \in \mathcal{L}(E)$ at x^{eq}. The spectral mapping theorem§ [?][Th.VII.3.11] states that the spectrum $\Sigma(e^{TA})$ of e^{TA} is given by $\{e^{T\lambda},\ \lambda \in \Sigma(A)\}$ which shows that we can apply Theorem 2.1. We let the reader complete the proof. ∎

2.2. Case of periodic orbits

To be done.

3. Tools for bifurcation theory

We recall that a linear mapping $A : Z \to X$ where Z, X are normed vectors spaces, is continuous iff $\|A\|_{\mathcal{L}(Z,X)} \equiv \sup\limits_{\|x\|_Z \leq 1} \|Ax\|_X < \infty$. We write $\mathcal{L}(Z, X)$ the set of continuous linear mappings between Z and X. We assume that $Z \subset X$ and we define the resolvent set

$$\rho(A) = \left\{\lambda \in \mathbb{C} \mid (\lambda Id - A) \textbf{ invertible and } (\lambda Id - A)^{-1} \in \mathcal{L}(X, X)\right\}$$

§ It is obvious in finite dimensions

and the spectrum

$$\Sigma(A) \stackrel{def}{=} \mathbb{C} \setminus \rho(A).$$

The *point spectrum* or the eigenvalues is the set $\Sigma_p(A) \subset \Sigma(A)$ defined by $\lambda \in \Sigma(A)$ such that $\ker(\lambda Id - A) \neq \{0\}$. In infinite dimensions, there are operators A for which $\Sigma_p(A) \subsetneq \Sigma(A)$.

We have not the space here to define the notion of *closed operator*, for which the spectral theory is well defined and studied. Indeed, one can show that if $\rho(A) \neq \emptyset$, then A is necessarily closed. The interested reader can look at [**?**, **?**]. We would like to mention that the hypothesis (iii) (see below) implies that A is closed in X.

Example: Let us consider $Au \stackrel{def}{=} \frac{d^2}{dx^2}u + u$ from $Z = \left\{u \in C^2([0,\pi]) \mid u(0) = u(\pi) = 0\right\}$ to $X = C^0([0,\pi])$. We investigate the spectrum of A. For this we have to solve the linear equation $\lambda u - Au = f$ for $\lambda \in \mathbb{C}, f \in X$ and $u \in Z$; that is, we have to find solutions $u \in C^2([0,\pi])$ of the linear problem $\lambda u - u - u'' = f$ with $u(0) = u(\pi) = 0$. The second order ODE has a unique solution $u \in C^2([0,\pi])$ satisfying the boundary conditions|| for $f \in C^0([0,\pi])$, precisely when the associated homogeneous equation $\lambda u - u - u'' = 0$ has no nontrivial solutions in X. A direct calculation shows that there are nontrivial solutions for $\lambda = 1 - n^2$, with n any positive integer. This shows that $\Sigma(A) = \left\{\lambda \in \mathbb{C} \mid \lambda = 1 - n^2,\ n \in \mathbb{N}^*\right\}$.

3.1. Center manifold

When the Jacobian at an equilibrium x^{eq} has an eigenvalue satisfying $\Re\lambda = 0$, hence breaking the stability condition of theorem 2.2, the vector field $F(x)$ around x^{eq} is particular. Indeed, there is a flow invariant manifold - also called invariant manifold - that contains all local bounded trajectories and the dimension of such manifold is equal to the sum of the algebraic multiplicities of the eigenvalues with zero real part. Hence, we can restrict the study of the dynamical system to this invariant manifold, which in the case of an infinite dimensional system, is very useful. More precisely, let us assume that $x^{eq} = 0$ and consider 3 Banach spaces X, Y, Z such that

$$Z \subset Y \subset X$$

with continuous embeddings, meaning the linear map $i : Y \to X$ (resp. $i : Z \to Y$) with $i(x) = x$ is continuous or equivalently $\|x\|_X \leq \|x\|_Y$ (resp. $\|x\|_Y \leq \|x\|_Z$). We write this as:

$$Z \hookrightarrow Y \hookrightarrow X.$$

We consider a differential equation of the form

$$\frac{dx}{dt} = F(x,\mu) \stackrel{def}{=} Ax + R(x,\mu) \tag{3}$$

where $\mu \in \mathbb{R}^m$ is a parameter. We assume the following, basically the jacobian of F is continuous at $(0,0)$ and the reminder is called R.

|| You can compute the solution analytically!

Hypothesis 3.1

(i) $A \in \mathcal{L}(Z, X)$

(ii) *for some* $k \geq 2$*, there exists a neighbourhood* $V_x \subset Z$ *of* 0 *and* V_μ *of* 0 *in* $\mathbb{R}^m$ *such that* $R \in C^k(V_x \times V_\mu, Y)$ *and* $R(0,0) = 0$*,* $d_x R(0,0) = 0$*.*

The reason why we chose such a general setting is to allow the study of more general differential equations. Indeed, it may occur, especially in infinite dimensions, that the domain of definition of the linear part differs from the one of the nonlinear part¶.

Hypothesis 3.2

(iii) *the spectrum of* A *is such that* $\Sigma(A) = \Sigma_- \cup \Sigma_+ \cup \Sigma_0$ *where* $\Sigma_- = \{\lambda \in \Sigma(A) \mid \Re\lambda < 0\}$*,* $\Sigma_- = \{\lambda \in \Sigma(A) \mid \Re\lambda > 0\}$ *and* $\Sigma_0 = \{\lambda \in \Sigma(A) \mid \Re\lambda = 0\}$*. Moreover the set* Σ_0 *consists of a* ***finite*** *number of eigenvalues with finite algebraic multiplicities. Finally, we require a* ***spectral gap*** *i.e. that there is* $\gamma > 0$ *verifying* $\inf_{\lambda\in\Sigma_+}(\Re\lambda) > \gamma$ *and* $\sup_{\lambda\in\Sigma_-}(\Re\lambda) < -\gamma$*.*

Note that this implies that $\Sigma \neq \emptyset$. Under these hypotheses, there is a unique spectral projector[?] $P_0 \in \mathcal{L}(X, Z) \cap \mathcal{L}(X, X)$ which commutes with A on Z. Defining $P_h \overset{def}{=} Id - P_0$, we write the ranges $E_0 \overset{def}{=} P_0 X = \ker(P_h) = \oplus_{\lambda\in\Sigma_0} E_\lambda(A)$ where $E_\lambda(A) = \oplus_{k\geq 0} \ker(\lambda Id - A)^k$ and $X_h \overset{def}{=} P_h X$. This allows to decompose X into invariant subspaces:

$$X = E_0 \oplus X_h, \;\; A \in \mathcal{L}(E_0), \;\; A \in \mathcal{L}(P_h Z, X_h).$$

It follows that $\Sigma(A|_{E_0}) = \Sigma_0$ and $\Sigma(A|_{P_h Z}) = \Sigma_+ \cup \Sigma_-$. Note that E_0 is called the **central** part and X_h the **hyperbolic** part.

We need another hypothesis before we can state the theorem of the center manifold. More precisely

Hypothesis 3.3

(iv) *For any* $\eta \in [0, \gamma]$ *and any* $f \in C_\eta(\mathbb{R}, P_h Y) \equiv \{f \in C^0(\mathbb{R}, P_h Y) \mid \|f\|_{C_\eta(\mathbb{R}, P_h Y)} \equiv \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|f(t)\|_Y < \infty\}$*, the linear problem*

$$\frac{du_h}{dt} = A|_{X_h} u_h + f(t) \tag{4}$$

has a unique solution $u_h = K_h f \in C_\eta(\mathbb{R}, P_h Z)$*. Furthermore, the linear map* K_h *belongs to* $\mathcal{L}(C_\eta(\mathbb{R}, P_h Y), C_\eta(\mathbb{R}, P_h Z))$*, and there exists a continuous map* $C : [0, \gamma] \to \mathbb{R}$ *such that* $\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R}, P_h Y), C_\eta(\mathbb{R}, P_h Z))} \leq C(\eta)$*.*

To check the hypothesis 3.3 is quite difficult in general. Hence, we use the following proposition.

¶ As in $\frac{dx}{dt} = \Delta x + x^2$ for example

Proposition 3.1

(i) *In finite dimension, i.e. when $X = \mathbb{R}^n$, hypothesis 3.3 is automatically satisfied.*

(ii) *In the infinite dimensions when $Y \neq X$, hypothesis 3.3 is satisfied if there exist positive constants $\omega_0 > 0, c > 0$ and $\alpha \in [0, 1)$ such that for all $\omega \in \mathbb{R}$, with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent set $\mathbb{C} \setminus \Sigma(A)$ of A, and*

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq c/|\omega|$$

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y,Z)} \leq c/|\omega|^{1-\alpha}$$

(iii) *If X, Y, Z are Hilbert spaces and $Y \neq X$, hypothesis 3.3 is satisfied if only the first inequality is valid in (ii).*

Proof. We only prove (i). In order to solve (4), the initial condition $u_h(0)$ is uniquely determined by the exponential growth required for the solution, $u_h \in C_\eta(\mathbb{R}, P_h Z)$, which is given by

$$u_h(t) = -\int_t^\infty e^{A(t-s)} P_+ f(s) + \int_{-\infty}^t e^{A(t-s)} P_- f(s) ds$$

where $P_\pm$ are the projectors associated to $\Sigma_\pm$. ■

We are now in position to give the center manifold theorem.

Theorem 3.2 *([?])*

There is a neighbourhood $\mathcal{O} = \mathcal{O}_x \times \mathcal{O}_\mu$ of $(0, 0)$ in $Z \times \mathbb{R}^m$, a mapping $\Psi \in C^k(E_0 \times \mathbb{R}^m; T^h)$ with

$$\Psi(0, 0) = 0, \quad d_x\Psi(0, 0) = 0$$

and a manifold $\mathcal{M}(\mu) = \{x_c + \Psi(x_c, \mu), x_c \in E_0\}$ for $\mu \in \mathcal{V}_\mu$ such that:

- *$\mathcal{M}(\mu)$ is **locally invariant**, i.e., $x(0) \in \mathcal{M}(\mu) \cap \mathcal{O}_x$ and $x(t) \in \mathcal{O}_x$ for all $t \in [0, T]$ implies $x(t) \in \mathcal{M}(\mu)$ for all $t \in [0, T]$.*
- *$\mathcal{M}(\mu)$ contains the set of **bounded solutions** of (3) staying in $\mathcal{O}_x$ for all $t \in \mathbb{R}$, i.e. if x is a solution of (1) satisfying for all $t \in \mathbb{R}$, $x(t) \in \mathcal{O}_x$, then $x(0) \in \mathcal{M}(\mu)$.*
- *(Parabolic case) if $\Sigma_+ = \emptyset$, then $\mathcal{M}(\mu)$ is **locally attracting**, i.e. if x is a solution of (1) with $x(0) \in \mathcal{O}_x$ and $x(t) \in \mathcal{O}_x$ for all $t > 0$, then there exist $v(0) \in \mathcal{M}(\mu) \cap \mathcal{O}_c$ and $\tilde{\gamma} > 0$ such that*

  $$x(t) = v(t) + O(e^{-\tilde{\gamma}t}) \text{ as } t \to \infty$$

  where v is a solution of (3) with initial condition $v(0)$.

We wish to make some remarks concerning the center manifold

- The Center manifold is not unique (cf. exercises)

- If $x_c(0) \in \mathcal{M}(\mu)$, then

$$\dot{x}_c = A|_{E_0} x_c + P_0 R(x_c + \Psi(x_c, \mu), \mu) \equiv f(x_c)$$

where P_0 is the (unique) projector on E_0 which commutes with A.

- The local coordinates function satisfies

$$d\Psi(x_c, \mu) \cdot f(x_c) = P_h A \cdot \Psi(x_c, \mu) + P_h R(x_c + \Psi(x_c, \mu))$$

- There are extensions for non-autonomous systems, DS with symmetries...
- Taylor expansion of Ψ is uniquely determined.
- You don't need to know what is a manifold (see differential geometry lectures for a proper definition) to understand this theorem. Suffice it to see that we describe the manifold $\mathcal{M}(\mu)$ here, as a small deviation Ψ from an hyperplane.

3.2. Normal form

The idea is to find a polynomial change of variables which *improves* locally a nonlinear system in **finite dimensional space** of the form

$$\dot{x} = Ax + R(x, \mu) \tag{5}$$

in order to analyse its dynamics more easily. This is usually used in conjunction to the center manifold. More precisely, the flow is reduced to the finite dimensional center (invariant) manifold in a first step and then simplified into the normal form in a second step. Note that there is a way to combine both operations in a single step (see [?]). We make the following hypothesis

Hypothesis 3.4

(i) $A \in \mathcal{L}(\mathbb{R}^n)$,

(ii) $R \in C^k(V_x \times V_\mu, \mathbb{R}^m)$ *and*

$$R(0,0) = 0, \; d_x R(0,0) = 0$$

Theorem 3.3

Assume the hypothesis hold. Then for any positive integer $p \in [2, k]$, there exist neighbourhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that for any $\mu \in \mathcal{V}_2$, there is a polynomial $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ of degree p with the following properties:

- *The coefficients of the monomials of degree q in Φ_μ are functions of μ of class C^{k-q}, and*

$$\Phi_0(0) = 0, \; d\Phi_0(0) = 0$$

- *For any $x \in \mathcal{V}_1$, the polynomial change of variable $x = y + \Phi_\mu(y)$ transforms (5) into the* ***normal form***

$$\dot{y} = Ay + N_\mu(y) + \rho(y, \mu)$$

and the following properties hold:

(i) For any $\mu \in \mathcal{V}_2$, N_μ is a polynomial $\mathbb{R}^n \to \mathbb{R}^n$ of degree p, with coefficients depending upon μ, such that the coefficients of the monomials of degree q are of class C^{k-q}, and

$$N_0(0) = 0, \; d_x N_0(0) = 0$$

(ii) the equality $\boxed{N_\mu(e^{tA^*}y) = e^{tA^*}N_\mu(y)}$ holds for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\mu \in \mathcal{V}_2$

(iii) the maps ρ belongs to $C^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and

$$\forall \mu \in \mathcal{V}_2, \; \rho(y, \mu) = o(\|y\|^p)$$

We note that N_μ is only polynomial in y. We can of course Taylor expand the monomials in μ to find there expression. It can be useful to use a relation that is equivalent[+] to the above equality (ii):

$$d_y N_\mu(y) A^* y = A^* N_\mu(y), \; \forall y \in \mathbb{R}^n, \; \mu \in \mathcal{V}_2.$$

3.2.1. Case of discrete DS We now state a version of the previous theorem in the case of discrete dynamical systems

$$x_{q+1} = Ax_q + R(x_q, \mu). \tag{6}$$

As above, we assume that $L \in \mathcal{L}(\mathbb{R}^n)$, $R \in C^k(\mathcal{V}_x \times \mathcal{V}_\mu, \mathbb{R}^m)$ and also that $R(0,0) = 0$, $d_x R(0,0) = 0$. It implies that $x = 0$ is an equilibrium for (6).

[+] It is easily proved by taking the differential in t at $t = 0$

Theorem 3.4 (*Normal form for discrete DS*)

Assume the hypothesis hold. Then for any positive integer $p \in [2, k]$, there exist neighbourhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that for any $\mu \in \mathcal{V}_2$, there is a polynomial $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ of degree p with the following properties:

- *The coefficients of the monomials of degree q in Φ_μ are functions of μ of class C^{k-q}, and*

$$\Phi_0(0) = 0, \ d\Phi_0(0) = 0$$

- *For any $x \in \mathcal{V}_1$, the polynomial change of variable $x = y + \Phi_\mu(y)$ transforms (6) into the normal form*

$$y_{q+1} = Ay_q + N_\mu(y_q) + \rho(y_q, \mu)$$

and the following properties hold:

(i) for any $\mu \in \mathcal{V}_2$, N_μ is a polynomial $\mathbb{R}^n \to \mathbb{R}^n$ of degree p, with coefficients depending upon μ, such that the coefficients of the monomials of degree q are of class C^{k-q}, and

$$N_0(0) = 0, \ d_x N_0(0) = 0$$

(ii) the equality $\boxed{N_\mu(A^* y) = A^* N_\mu(y)}$ holds for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\mu \in \mathcal{V}_2$

(iii) the maps ρ belongs to $C^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and

$$\forall \mu \in \mathcal{V}_2, \ \rho(y, \mu) = o(\|y\|^p)$$

4. Basic bifurcations

We assume that the model has been written on the center manifold in the following way

$$\dot{u} = f(u, \mu).$$

A basic question concerns the stability of equilibria. A useful result in this direction is Th. 2.2. This theorem breaks down when they are some eigenvalues λ such that $\Re\lambda$. This can happen in two ways:

- 0 is an eigenvalue
- $\pm i\omega$ is an eigenvalue,

in which case an invariant manifold, the center manifold exists. For the first condition to occur, a scalar system is enough. For the second case, a 2d system is necessary. The saddle node bifurcation corresponds to the first case (see theorem below with parameter $\mu = 0$). Basically, it predicts the birth of additional equilibria.

The two following cases are standart:

Theorem 4.1 (*Saddle-Node bifurcation*)

*Assume f is **scalar** $C^k, k \geq 2$ in a neighborhood of $(0,0)$, and that it satisfies*

$$f(0,0) = 0, \ \frac{\partial}{\partial u} f(0,0) = 0$$

and

$$\frac{\partial}{\partial \mu} f(0,0) := a \neq 0, \frac{\partial^2}{\partial^2 u} f(0,0) := 2b \neq 0.$$

*Then, **a saddle-node bifurcation** occurs at $\mu = 0$. More precisely, the following properties hold in a neighbourhood of 0 in $\mathbb{R}$ for sufficiently small μ:*

- *if $ab < 0$ (resp. $ab > 0$) the differential equation has 2 equilibria $u_\pm(\epsilon), \epsilon = \sqrt{\mu}$ for $\mu > 0$ (resp., for $\mu < 0$) , with opposite stabilities. Furthermore, the map $\epsilon \to u_\pm(\epsilon)$ is of class C^{k2} in a neighbourhood of 0, and $u_\pm(\epsilon) = O(\epsilon)$.*
- *if $ab < 0$ (resp. $ab > 0$) the differential equation has no equilibria for $\mu < 0$ (resp., for $\mu > 0$).*

The proof of the above theorem has been proposed in exercises. It amounts to first study the vector field truncated at order 2. It gives a simple scalar system to analyse, namely $\dot{u} = a\mu + bu^2$. Then, one has to show that the dynamics persists when one consider the full system $\dot{u} = f(u, \mu)$.

For the second case, the system has an eigenvalue $\pm i\omega$ for a specific value of the parameter μ (also called a bifurcation parameter). This gives a 2d system on the center manifold. Writing such a vector field is best done in complex coordinates, one finds $\dot{z} = f(z, \bar{z}, \mu)$. We can then appeal to the normal form theorem to simplify / transform this complex ODE, by mean of a change of variable into a *normal form* which can be shown to be given by the equation below $\dot{A} = (\mu + i\omega)A + \cdots$.

Theorem 4.2 (*Hopf bifurcation*)

Assume f is $C^k, k \geq 5$ in a neighbourhood of $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$, and that it satisfies

$$f(0,0) = 0, \;\; L := \partial_u f(0,0) = 0.$$

Assume that the two eigenvalues of the linear operator L are $\pm i\omega$ for some $\omega > 0$. Finally, assume that the normal form can be written

$$\frac{d}{dt}A = (\mu + i\omega)A + b|A|^2 A + O((|\mu| + |A|^2)^2)$$

with $\Re a, \Re b \neq 0$. Then, **a Hopf bifurcation** *occurs at $\mu = 0$. More precisely, the following properties hold in a neighbourhood of 0 in $\mathbb{R}^2$ for sufficiently small μ:*

- *if $\Re a \Re b < 0$ (resp. $ab > 0$) the differential equation has precisely one equilibrium $u(\mu)$ for $\mu < 0$ (resp., for $\mu > 0$), with $u(0) = 0$. This equilibrium is stable when $\Re b < 0$ and unstable when $\Re b > 0$.*
- *if $\Re a \Re b < 0$ (resp. $\Re a \Re b > 0$) the differential equation possesses for $\mu > 0$ (resp., for $\mu < 0$) an equilibrium $u(\mu)$ and a unique periodic orbit $u^*(\mu) = O(\sqrt{|\mu|})$, which surrounds this equilibrium. The periodic orbit is stable when $\Re b < 0$ and unstable when $\Re b > 0$, whereas the equilibrium has opposite stability.*

The proof of the above theorem was proposed in an exercise. One first look at the truncated system by means of polar coordinates and then proves that the observed periodic orbit persists for the full system.

Appendix A. Jordan decomposition

Contents

References