- Draft - Summary for the MMN lectures

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1. Picard theorems

We start with a very standard result that is the basis for most results in these notes. This first section is based on [Viterbo, 2003].

1.1. Basic results

Definition 1.1

A function $T : X \to X$ where (X, d) is a metric space is contracting iff $\forall x, y \in X$ $d(Tx, Ty) \leq k \cdot d(x, y)$ where $k < 1$.

Theorem 1.1 (*Picard*)

Let X be a complete metric space and $T : X \to X$ a contracting mapping. Then T has a unique fixed point i.e. the unique solution of the equation $T(y) - y = 0$.

Proof. Take $x_0 \in X$ and show that $(T^n x_0)_n$ is a Cauchy sequence hence convergent (in a complete space), its limit will be the solution of the fixed point equation. If $n \geq m$

$$d(T^n x_0, T^m x_0) \leq d(T^m(T^{n-m}x_0), T^m x_0) \leq k^m d(T^{n-m}x_0, x_0).$$

We have from the triangle inequality:

$$d(T^{n-m}x_0, x_0) \leq \sum_{i=0}^{n-m-1} d(T^{i+1}x_0, T^i x_0) \leq \sum_{i=0}^{n-m-1} k^i d(Tx_0, x_0)$$
$$\leq \frac{1-k^{n-m}}{1-k} d(Tx_0, x_0) \leq \frac{1}{1-k} d(Tx_0, x_0).$$

This gives:

$$d(T^m x_0, T^n x_0) \leq \frac{k^m}{1-k} d(Tx_0, x_0)$$

which shows that $(T^n x_0)_n$ is a Cauchy sequence. ∎

We provide without proof a version with parameters. To have a notion of differentiability, we need to assume that we work in a Banach space. Hence, if $p = 0$, X is a metric space and a closed subspace of a Banach space if $p > 0$. Also, if $p = 0$,

Λ is a metric space and an *open* subspace of a Banach space if $p > 0$. We consider a mapping $T \in C^p(X \times \Lambda, X)$ such that $x \to T(x, \lambda)$ is Lipschitz with constant $k(\lambda)$ where $\lambda \to k(\lambda)$ is continuous. We say that we have a C^p **map of contracting functions** if moreover $k(\lambda) < 1$.

Theorem 1.2 (*Picard with parameters*)

A C^p map of contracting functions on a space X has a family of fixed points $x(\lambda)$ where the map $\lambda \to x(\lambda)$ is C^p.

1.2. Applications, see [Chow and Hale, 1982]

We work here in Banach spaces, *i.e.* complete normed vector space. Recall that a linear map $L \in \mathcal{L}(E, F)$ between Banach spaces is continuous iff $\|L\|_{\mathcal{L}(E,F)} \equiv \sup_{\|x\|_E \leq 1} \|Lx\|_F < \infty$.

Theorem 1.3 (*Implicit functions theorem*)

*Let us consider two open sets U, V in Banach spaces E_1, E_2 and $f : U \times V \to F$ a C^k application with $k \geq 1$. We assume $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \in \mathcal{L}(E_2, F)$ has a **continuous** inverse. Then there are neighbourhoods U' of x_0, V' of y_0 and a mapping $\phi \in C^k(U', V')$ such that*

$$\forall (x, y) \in U' \times V',\ f(x, y) = 0 \Leftrightarrow y = \phi(x).$$

Proof. For‡ $x \in B(x_0, r)$ and $y \in B(y_0, r')$, the mapping

$$T_x(y) = y - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} f(x, y)$$

is continuous from $\overline{B(y_0, r')}$ to itself if r, r' are small enough. Indeed:

$$dT_x(y) = Id - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} \frac{\partial f}{\partial y} f(x, y)$$

is close to zero hence we can chose r, r' small enough so that for all x in $B(x_0, r)$ and y in $B(y_0, r')$, we have $\|dT_x(y)\| \leq \frac{1}{2}$. Then it gives $\|T_x(y_0) - y_0\| = \| \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} f(x, y)\| \leq r'/2$ if r, r' are small enough. It follows that $T_x\left(\overline{B(y_0, r')}\right) \subset \overline{B(T_x(y_0), r'/2)} \subset \overline{B(y_0, r')}$ (strict inclusion). Hence, $T_x : \overline{B(y_0, r')} \to \overline{B(y_0, r')}$ is contracting and we conclude with the Picard theorem with parameters. ∎

We also have a result to invert a nonlinear map.

‡ We use the definition $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$

Theorem 1.4 (*Inverse function theorem*)

Let $\phi \in C^k(U,V)$ and $d\phi(x_0)$ has a bounded inverse. There are open sets U', V' containing x_0 and $y_0 = \phi(x_0)$ and a map $\psi \in C^k(V', U')$ such that $\phi \circ \psi = Id$.

Proof. We apply the previous theorem to $f(u,v) = v - \phi(v)$. ■

1.3. Cauchy-Lipschitz theorems

We consider $F : I \times \Omega \to E$ where Ω is an open set of a Banach space E and I is an open interval of $\mathbb{R}$. We want to solve

$$\dot{x} = F(t,x) \tag{1}$$

with the initial condition $x(t_0) = x_0 \in \Omega, t_0 \in I$.

Theorem 1.5 (*Cauchy-Lipschitz*)

We assume that F is continuous and Lipschitz in the second variable. Then for all $\tau \in I$ and $u_0 \in \Omega$ there exist $\delta, \alpha > 0$ such that the system

$$\begin{cases} \dot{x} = F(t,x), \\ x(t_0) = x_0 \in \Omega \end{cases}$$

has a unique solution defined on $]t_0 - \alpha, t_0 + \alpha[$ for all $x_0 \in B(u_0)$ and $t_0 \in]\tau - \delta, \tau + \delta[$.

Proof. Left in exercise, you may use the Picard theorem. ■

The theorem looks complicated in this form because we want that for t_0 in a neighbourhood of τ and x_0 in a neighborhood of u_0, the size 2α of the domain of definition is uniformly minored. We also give a version with parameters. Note that the initial condition can be taken as a parameter.

Theorem 1.6 (*Cauchy-Lipschitz with parameters*)

We assume that $F(\lambda, t, x)$ is in $C^k(\Lambda \times I \times \Omega, E)$ with $k \geq 0$, Lipschitz in the variable x and Λ satisfies the hypothesis of theorem 1.2. Let $x_\lambda(t; t_0, x_0)$ be the solution of

$$\begin{cases} \dot{x} = F(\lambda, t, x), \\ x(t_0) = x_0 \in \Omega \end{cases}$$

Then for all $(\lambda_0, \tau, u_0) \in \Lambda \times I \times \Omega$, there exist $\delta, \alpha > 0$ such that x_λ is defined on $]t_0 - \alpha, t_0 + \alpha[$ for all $(\lambda_0, t_0, x_0) \in B(\lambda_0, \delta) \times]\tau - \delta, \tau + \delta[\times B(u_0, \delta)$. Moreover, the map $(t, \lambda, t_0, x_0) \to x_\lambda(t; t_0, x_0)$ is C^k.

Up to the domain of definition, this theorem amounts to saying that $(t, \lambda, t_0, x_0) \to x_\lambda(t; t_0, x_0)$ is C^k.

An important question is to determine whether the solution exists on I entirely. It turns out that this is not possible if “the solution explodes in finite time”.

Proposition 1.7

Let $F : (t_-, t_+) \times \Omega \to E$ satisfying the conditions of Theorem 1.5 and x be a maximal solution defined on $J = (\tau_-, \tau_+)$. If $\tau_+ < t_+$ then $x(t)$ leaves every compact as t tends to τ_+.

When E is finite dimensional and $\Omega = E$, the conclusion is equivalent to $\lim_{t \to \tau_+} \|x(t)\| = \infty$. It follows that if we can bound the solution by a finite quantity, then the solution is defined on I. This is one example of use of the following lemma.

Lemma 1.1 (*Gronwall*)

Let $g(t, r)$ be defined on $I \times \mathbb{R}$, Lipschitz in the second variable and ρ is the unique solution, assumed to be defined on I, of

$$\begin{cases} \dot{\rho} = g(t, \rho), \\ \rho(t_0) = \rho_0. \end{cases}$$

Then, if $x : I \to \mathbb{R}^n$ (finite dimension!!) satisfies $\|\dot{x}(t)\| \leq g(t, \|x(t)\|)$ and $\|x(t_0)\| \leq \rho_0$, then $\forall t \geq t_0$ in I, we have $\|x(t)\| \leq \rho(t)$.

Hence, in finite dimensions, if $\|X(t, x)\| \leq g(t, \|x\|)$, then the maximal interval of existence of x contains the one of ρ.

For example, in finite dimensions, if $\|X(t, x)\| \leq A\|x\| + B$, then the solutions are defined on I, no finite time explosion occurs.

Contents

References

[Chow and Hale, 1982] Chow, S.-N. and Hale, J. K. (1982). *Methods of Bifurcation Theory*, volume 251 of *Grundlehren der mathematischen Wissenschaften*. Springer New York, New York, NY.

[Viterbo, 2003] Viterbo, C. (2003). *Equations differentielles et systemes dynamiques*. Ecole Polytechnique.