ENS - Master MVA / Paris 6 - Master Maths-Bio

Tutorial 1

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Do not hesitate to contact me for more details concerning the solutions.

Tut: done during the tutorials

Exercise

Ultra basic

- (**Tut**) Find the solution of $\dot{x} = Ax + b(t)$ where $A \in M_n(\mathbb{R})$ and $b \in C^0(\mathbb{R}, \mathbb{R}^n)$.

> $x(t) = e^{tA}x_0 + \int_0^r e^{(t-s)A}b(s)ds$

- If $x(t) \leq f(t) + \int_0^t h(s)x(s)ds$ with x, f, g continuous and $f, h \geq 0$, show that

$$x(t) \leq f(t) + \int_0^t f(s)h(s)\exp\left(\int_s^t h(r)\mathrm{d}r\right)\mathrm{d}s.$$

> Define $X(t) = \int_0^t hx$ and differentiate it to obtain $\frac{d}{dt}\left[e^{-\int_0^t h}X(t)\right] \leq h(t)f(t)e^{-\int_0^t h}$ and then integrate this formula.

Exercise

Around the Cauchy-Lipschitz theorem

- (**Tut**) (uniqueness) Using the Bernouilli principle, one can show that the water height of a tank with leaking hole at the bottom satisfies the following equation

$$\dot{h}(t) = -A\sqrt{h(t)}$$

where A is a positive constant depending on the physical parameters of the problem. Show that there are multiple solutions to this equation satisfying the condition $h(T) = 0$ for $T > 0$. Knowing the state of the tank at time t_0, can we find the tank state at any time?

> The function $h(t) = \frac{A^2}{4}(T-t)^2$ if $t \leq T$ and 0 otherwise is a solution. $h = 0$ is another solution, the non uniqueness comes from $x \to \sqrt{x}$ not being Lip at 0.

- Let us consider the globally defined flow $t \to \phi^t(x)$ and the map $F : x \to \phi^T(x)$. Assume that x^* is a fixed point of F. Show that $\phi^s(x^*)$ is also a fixed point of F. Deduce that if x^* is an isolated fixed point of F, then it is an equilibrium point.

> $F(\phi^s(x^*)) = \phi^{T+s}(x^*) = \phi^s(F(x^*)) = \phi^s(x^*)$. If x^* is isolated, then $\phi^t(x^*) = x^*$ for t small enough: x^* is an equilibrium.

- Assume that there is an invariant circle for F, $\Gamma = \{\theta \to X_0(\theta),\ \theta \in \mathbb{R}/\mathbb{Z}\}$. Show that $\Gamma_s = \{\theta \to \phi^s(X_0(\theta)),\ \theta \in \mathbb{R}/\mathbb{Z}\}$ is also an invariant circle for F. We assume that X_0 is C^1 and $X_0'(\theta) \neq 0$ for all θ. Deduce that if Γ is an isolated invariant circle for F, then Γ is a trajectory of the differential equation.

> $F(\Gamma_s) \subset \Gamma_s$ as in previous question. We write
> $$\phi^s(X_0(\theta)) = X_0(f(\theta, s)) \tag{E}$$
> for s small enough. We note that $f(\theta, s) = \theta + S(\theta, s)$ where $S(\theta, s) = o(s)$. f is C^1 by the Implicit functions theorem. Differentiating (E) en $s = 0$ gives
> $F(X_0(\theta)) = X_0'(\theta)S'(\theta, 0)$ and so Γ is a trajectory of the differential equation.

Exercise

Existence theorem for HH equations (Tut)

We consider the HH equations

$$C\dot{V} = I - \bar{g}_K n^4(V - E_K) - \bar{g}_{Na} m^3 h(V - E_{Na}) - \bar{g}_L(V - E_L)$$

$$\dot{x} = \frac{1}{\tau_x(V)}(x_\infty(V) - x), \quad x \in \{m, n, h\}$$

Show that the solution is defined on $\mathbb{R}^+$.

> The gating variables x describe the proportion of open channels, hence $x \in [0,1]$. It follows that the vector field is bounded as follows: $|\dot{V}| \leq A|V| + B$ where $A, B > 0$. It is a well known result that the flow is defined on $\mathbb{R}^+$, *e.g.* by using Gronwall lemma (to show the non-explosion of the flow).

Exercise

Firing rate of Integrate and Fire (IF) neurons

- We consider the following model of spiking neuron

$$\tau \frac{dV}{dt} = E_L - V + I$$

with threshold θ and reset V_r. Find the condition which ensures spiking. Find the firing rate as function of I.

> $V(t) = E_L + I + e^{-t/\tau}(V_0 - E_L - I)$ hence the condition $E_L + I > \theta$. The firing rate f satisfies $V(1/f) = \theta$ with $V_0 = V_r$. One finds $f = \frac{1}{\tau}\log\left(\frac{E_L+I-V_r}{E_L+I-\theta}\right)$.

Exercise

The θ-neuron (Tut)

It is an abstract model of spike generation. The potential is described by $\theta \in \mathcal{S}^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$ and satisfies:

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I$$

where I is the injected current. We consider that a spike is emitted when θ crosses the point $\theta = \pi$.

1. Show that for $I < 0$, there are two equilibria for the system, one stable and the other unstable. Show that every solution not starting at the unstable equilibrium converge to the stable equilibrium.
2. In the case $I > 0$, show that there is no equilibrium. Conclude that the trajectories are periodic orbits with regular spiking.
3. What happens when $I = 0$?

> 1) We find $\theta_{\pm} = \pm acos\frac{1+I}{1-I}$ hence producing two solutions $\theta_{\pm}$. Writing the equation as $\dot{\theta} = F(\theta)$, the (nonlinear) stability of the points is given by the sign of the real part of the eigenvalues of $dF(\theta_{\pm})$. One finds a single eigenvalue $\lambda_{\pm} = \pm\sqrt{-I}$. Hence θ_{+} is unstable. In between those points, the vector field does not vanish, this is used to show the convergence to the fixed point θ_{-}. 2) similar to 1). 3) There is a saddle-node bifurcation, the unique equilibrium $\theta = 0$ is unstable.

Exercise

Saddle node bifurcation, *cf* previous exercise

Assume f is a scalar $C^k, k \geq 2$ map in a neighborhood of $(0, 0)$, and that it satisfies

$$f(0,0) = 0,\ \frac{\partial}{\partial u}f(0,0) = 0$$

and

$$\frac{\partial}{\partial \mu}f(0,0) := a \neq 0, \frac{\partial^2}{\partial^2 u}f(0,0) := 2b \neq 0$$

where μ is a parameter. Then, a *saddle-node bifurcation* occurs at $\mu = 0$. More precisely, the following properties hold in a neighborhood of 0 in $\mathbb{R}$ for sufficiently small μ:

- if $ab < 0$ (resp. $ab > 0$) the differential equation has 2 equilibria $u_{\pm}(\epsilon), \epsilon = \sqrt{|\mu|}$ for $\mu > 0$ (resp., for $\mu < 0$), with opposite stabilities. Furthermore, the map $\epsilon \to u_{\pm}(\epsilon)$ is of class C^{k-2} in a neighborhood of 0, and $u_{\pm}(\epsilon) = O(\epsilon)$.
- if $ab < 0$ (resp. $ab > 0$) the differential equation has no equilibria for $\mu < 0$ (resp., for $\mu > 0$).

1. Write the Taylor expansion at order 2 for (u, μ) close to zero.
2. Study the truncated equation at order 2. Find the equilibria and their stability. Plot the equilibria as function of the parameter μ. The value $\mu = 0$ is called a **bifurcation point**.
3. Extend the results to the case where the reminder is not neglected anymore.

> 1) $f(u, \mu) = a\mu + b\mu^2 + o(|\mu| + u^2)$. 2) We look at $\dot{u} = a\mu + b\mu^2$. The equilibria are solutions of $a\mu + b\mu^2 = 0$. The rest follows like for the Exercise θ-neuron.

Exercise

Hopf bifurcation, birth of (spontaneous) oscillations

We consider the equation

$$\dot{z} = z(i + a - |z|^2) \tag{E}$$

where $a \in \mathbb{R}$

1. Analyse the stability of equilibria.
2. Show that when $a > 0$, the is a (stable) periodic orbit.
3. (*Difficult*) Extend to the case $\dot{z} = f(z, \bar{z}, a)$ where the righthand side of (E) is the Taylor expansion of f at order 3.