ENS - Master MVA / Paris 6 - Master Maths-Bio

Tutorial 2

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Exercice 1

Non-uniqueness of the center manifold

Consider the system

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y. \end{cases}$$

Show that $\mathcal{M}(\beta) = \{(x, y) \mid y = A(x)\}$ with $A(x) = \beta e^{\frac{1}{x}}$ si $x < 0$ and $A(x) = 0$ otherwise, is a family of center manifolds.

(Hint) You may write the equation satisfied by the center manifold.

> $(0, 0)$ is an equilibrium. The Jacobian at this equilibrium is $J = diag(0, -1)$. Hence, the linear center space associated to the eigenvalue(s) of zero real part is spanned by $(1, 0)$. The center manifold theorem states that there is an invariant manifold $\{u \cdot (1, 0) + (0, A(u))\} = \{(u, A(u))\}$. The manifold is invariant meaning that if $(x_0, y_0) = (u_0, A(u_0))$ for some u_0 , then for all $t \geq 0$, $(x(t), y(t)) = (u(t), A(u(t)))$. It gives $x(t) = u(t)$ and $y(t) = A(x(t))$. We differentiate *w.r.t.* time and get $x(t)^2 A'(x(t)) = -A(x(t))$. If the manifold is defined over an open set, we can evaluate the previous equation at $t = 0$ and this gives $x^2 A'(x) = -A(x)$ on an open set. Solving this ODE with the conditions $A(0) = A'(0) = 0$ gives the solution. Reciprocally, we check directly that the manifold is indeed invariant by the dynamics and tangent to $(0, 0)$.

Exercice 2

Analytic center manifold

Show that the following system

$$\dot{x} = -x^3$$
$$\dot{y} = -y + x^2$$

does not have an analytic centre manifold.

> Suppose that one has a centre manifold y = h(x), where h is analytic at x = 0. Then $h(x) = \sum_{n=2}^{\infty} a_n x^n$ for small x and $h'(x)x^3 = h(x) - x^2$. One can show that $a_{2n+1} = 0$ for all n and that $a_{n+2} = na_n$ for n = 2,4,..., with $a_2 = 1$. The convergence radius is then zero.

Exercice 3

Slow-Fast systems

Consider the general system

$$(E_\epsilon) : \begin{cases} \dot{x} = & f(x, y, \epsilon) \\ \dot{y} = & \epsilon g(x, y, \epsilon) \end{cases} \qquad \left(\text{recall Fitzhugh-Nagumo} \begin{cases} \dot{v} = & v - \frac{v^3}{3} - w + I \\ \dot{w} = & \epsilon\,(v + a - bw) \end{cases}\right)$$

The *critical manifold* is defined by $\mathcal{S} = \{(x, y) \mid f(x, y, 0) = 0\}$. It corresponds to a set of equilibria for the *layer* problem (E_0).

1. Give a sufficient condition for the existence of a continuous function $h : \mathcal{D}_0 \to \mathbb{R}^m$ with $\mathcal{D}_0$ connected having a non-empty interior such that $\mathcal{M} \equiv \{(h(y), y),\ y \in \mathcal{D}_0\} \subset \mathcal{S}$. $\mathcal{M}$ is called a *slow manifold*.
2. We assume that $\mathcal{M}$ is uniformly hyperbolic. That is: $\exists \sigma > 0$ such that for all $y \in \mathcal{D}_0$, the eigenvalues are bounded away from zero: $\max_{\lambda \in \Sigma(\partial_x f(h(y), y))} |\Re \lambda| > \sigma$. Prove that there exists for ϵ small enough, a locally invariant manifold close to $(h(y_0), y_0)$ for some $y_0 \in \mathcal{D}_0$:

   $$\mathcal{M}_\epsilon = \{(x, y) : x = h(y, \epsilon),\ y \in \mathcal{V}(y_0)\}$$

   where $h(y, \epsilon) = h(y) + \mathcal{O}(\epsilon)$.
3. (difficult) Show that $\mathcal{M}_\epsilon$ is uniformly asymptotically stable if $\mathcal{M}$ is.

> 1) Implicit functions theorem. 2) Application of the center manifold theorem where the parameter is taken to be ϵ.

Exercice 4

Adaptive exponential integrate-and-fire model (AdExp)

We consider the model

$$\begin{cases} C\dot{v} = -g_L(v - E_L) + g_L \Delta_T \exp\left(\frac{v - V_T}{\Delta_T}\right) - w + I \\ \tau_w \dot{w} = a(bv - w). \end{cases}$$

where $g_L, a, b > 0$.

When the membrane potential v is high enough, the trajectory quickly diverges because of the exponential term. We call a spike a part of the trajectory with $V \approx \infty$. When a spike occurs (at an explosion time for the ODE), the membrane potential is instantaneously reset to some value v_r and the adaptation current is increased:

$$\begin{cases} v \to v_r \\ w \to w + d. \end{cases}$$

For simplicity, we restrict to the case $C = \tau_w = 1$. We also assume that F is (at least) C^3 and is strictly convex with $\lim_{-\infty} F = \infty$ and $\lim_{+\infty} F' = +\infty$ and $\lim_{-\infty} F' < 0$. Hence, we look at the model:

$$\begin{cases} \dot{v} = F(v) - w + I \\ \dot{w} = a(bv - w). \end{cases} \tag{1}$$

1. Draw the nullclines. Discuss the dynamics according to the initial condition.
2. Write the equations satisfied by the equilibria.
3. Write $G_b(v) = F(v) - bv$. Show that G_b is strictly convex with a unique minimum $m(b)$ that is attained for $v = v^*(b)$
4. Show that m, v^* are at least C^2.
5. Compute the number of equilibria and their stability as function of I and b. In particular, show that:

    1. For $I > -m(b)$, there are no equilibria
    2. For $I = -m(b)$, there is a unique equilibrium $(v^*(b), w^*(b))$ which is not hyperbolic. It is unstable if $b > a$.

3. If $I < -m(b)$, there are two equilibria $\left(v_-(I,b), v_+(I,b)\right)$ such that

$$v_-(I,b) < v^*(b) < v_+(I,b)$$

The equilibrium $v_+(I,b)$ is a saddle fixed point and the stability of $v_-(I,b)$ depends on I and $sign(b-a)$. If $b < a$ then $v_-(I,b)$ is attracting. If $b > a$ there is a smooth curve $I^*(a,b)$ defined implicitly by $F'\left(v_-(I^*(a,b)), b\right) = a$ such that if $I < I^*(a,b)$ then the equilibrium is attracting and if $I > I^*(a,b)$ then the equilibrium is repulsive.

6. Show that the curve $\{(b,I) \mid I = -m(b), a \neq b\}$ is a saddle-node bifurcation curve (when $F''(v^*) \neq 0$). Give the equation on the center manifold.
7. Assume $a > b$ and write v_a the unique solution of $F'(v_a) = a$. If $F''(v_a) \neq 0$, show that there is a Andronov-Hopf bifurcation at v_a on the curve $AH = \{(b,I); b > a \text{ and } I = bv_a - F(v_a)\}$.
8. Take $a > 0, b = a$ and assume that $F'''(v_a) \neq 0$. Show that the system has a Bogdanov-Takens bifurcation.

> 3) $w = bv, F(v) - bv + I = G_b(v) + I = 0$, 3) G_b is strictly convex and $G_b \overset{\pm\infty}{\to} +\infty$ shows that G_b has a (unique) minimum. 4) Apply implicit function theorem to $A(v,b) = f'(v) - b$. 5) If $I > -m(b)$ then $F(v) - bv + I > 0$ hence no equilibrium. For $I = -m(b)$, $v \to F(v) - bv + I$ vanishes only at $v^*(b)$, thus $L(v) = \begin{bmatrix} b & -1 \\ ab & -a \end{bmatrix}$ is non hyperbolic. If $I < -m(b)$ then $F(v^*) - bv^* + I < F(v^*) - bv^* - m(b) = 0$ and $F(v) - bv \overset{v\to+\infty}{\to} +\infty$ implies that there is $v_+(I,B) > v^*(b)$ which is an equilibrium. Idem for v_- from $F(v) - bv \overset{v\to-\infty}{\to} +\infty$. There can't be 3 fixed points because G_b is strictly convex (one can use the definition of a strictly convex function $f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$). Recall that $\det L(v^*) = 0$ and $\det L(v) = a(b - F'(v))$ is deacresing in v. $\det L(v_+) < 0$ shows that v_+ is a saddle. $\det L(v_-) > 0$, the trace gives the sign of the eigenvalues $trace(L(v_-)) = F'(v_-) - a < F'(v^*) - a = b - a$. Hence, if $b - a < 0$, then v_- is attracting. In the case $b > a$, let us define $A(I,b,a) = F'(v_-(I,b)) - a$. We have $\lim_{I\to-m(b)} A(I,a,b) = b - a > 0$ and $\lim_{I\to-\infty} A(I,a,b) = \lim_{v\to-\infty} = F'(v) - a < 0$. Plus $I \to v_-(I,b)$ is increasing, so there exists a curve $I^*(a,b)$ such that for $I^*(a,b) < I < -m(b)$, the fixed point $v_-(I,b)$ is repulsive and for $I < I^*(a,b)$, the fixed point v_- is attracting.
>
> 6) Let us write the ODE (1) as $\dot{V} = RHS(V, par_{sn} + \mu)$ where $\mu \approx 0$. We write par_{sn} the parameter values at the SN bifurcation point. The jacobian $L_0 \equiv L(v^*)$ has eigenvalues $(0, b-a)$. We write $L_0\zeta = 0$ with $\zeta = [1/b, 1]$ and $L_0^*\zeta^* = 0$ with $\zeta^* = [-a, 1]$. We

associate the following spectral projector P_0, on the kernel of L_0 - the center part-, which commutes with $L_0 : P_0 U =< \zeta^*, U > \zeta$. We assume that the SN bifurcation occurs for parameters labelled with $*_{sn}$ like $I_{sn}, \cdots$

The equation on the center manifold is now studied. To this end, we write (1) as $\dot U = L_0 U + R(U,\mu), R(0,0) = 0, dR(0,0) = 0$ where $(v,w) = (v^*, w^*) + U$, $L_0 = L(v^*)$ and $R(U,\mu) = RHS((v^*, bv^*) + U, \mu) - L_0 U$. On the center manifold, we have $U = A\zeta + \Psi(A,\mu)$ which gives $\dot A = f(A,\mu)$ where $f(A,\mu) = P_0 R(A\zeta + \Psi(A,\mu),\mu)$. Recall that $\Psi(0,0) = 0$ and $\partial_A \Psi(0,0) = 0$. Hence $\Psi(A,\mu) = O(\mu + A\cdot(\mu + A))$.

In view of the Saddle-Node bifurcation theorem, we have to find the coefficients of the Taylor expansion $f(A,\mu) = \alpha(I - I_{sn}) + \beta A^2 + o\left(|I - I_{sn}| + A^2\right)$. (Note that the term $(I - I_{sn})\cdot A$ is contained in the $o()$, indeed we do not need it).

Using a Taylor expansion of f, we first get $\alpha = \langle \zeta^*, \partial_I RHS(v^*, w^*)\rangle = \langle \zeta^*, [1,0]\rangle = -a < 0$. As R and Ψ are quadratic, one gets $\beta \propto \langle \zeta^*, d^2 RHS(v^*, w^*)[\zeta,\zeta]\rangle = -\frac{a}{b^2} F''(v^*) \neq 0$. These are the conditions for the application of the Saddle-node bifurcation.

7) If $a > b$, $F'(v_a) = a$ has a unique solution. The sufficient conditions are $Trace(L) = 0 = F'(v_H) - a$ and $0 < \det L = a(b - F'(v_H))$. Hence, we need $I = bv_a - F(v_a)$ to have two complex purely imaginary eigenvalues $\lambda(I)$. Let us check that $\partial_I \Re\lambda(I) \neq 0$. This is related to $\frac{1}{2}\partial_I trace(L(v_H)) = \frac{1}{2}F''(v_H)\partial_I v_-(I,b)$. We conclude with $\partial_I v_= \frac{1}{b - F'(v_-)} = \frac{1}{b-a} > 0$. The Lyapunov coefficient is more involved...

Exercice 5

Normal Form

The idea is to find a polynomial change of variable which simplifies locally a nonlinear system, by removing the maximum number of monomials, this in order to analyze its dynamics more easily. We thus consider a differential equation:

$$\dot x = \mathbf{L}x + \mathbf{R}(x;\alpha),\ \mathbf{L} \in \mathcal{L}(\mathbb{R}^n),\ \mathbf{R} \in C^k(\mathcal{V}_x \times \mathcal{V}_\alpha, \mathbb{R}^m) \quad (1)$$

$$\mathbf{R}(0;0) = 0,\ d\mathbf{R}(0;0) = 0$$

The normal form **Theorem** is the following:

> Then, $\forall p \in [2, k]$, there are neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that for any $\alpha \in \mathcal{V}_2$, there is a polynomial $\Phi_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ of degree p with the following properties:
>
> 1. The coefficients of the monomials of degree q in Φ_α are functions of α of class C^{k-q}, and
>
> $$\Phi_0(0) = 0,\ d\Phi_0(0) = 0$$
>
> 2. For any $x \in \mathcal{V}_1$, the polynomial **Change of Variable** $x = y + \Phi_\alpha(y)$ transforms (1) into the normal form
>
> $$\dot{y} = \mathbf{L}y + \mathbf{N}_\alpha(y) + \rho(y, \alpha)$$
>
> where $\mathbf{N}_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomials of degree p
> 3. The coefficients of the monomials of degree q in $\mathbf{N}_\alpha$ are functions of α of class C^{k-q}, and
>
> $$\mathbf{N}_0(0) = 0,\ d_x\mathbf{N}_0(0) = 0$$
>
> 4. the equality $\mathbf{N}_\alpha(e^{t\mathbf{L}^*}y) = e^{t\mathbf{L}^*}\mathbf{N}_\alpha(y)$ holds for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in \mathcal{V}_2$
> 5. the maps ρ belongs to $C^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and $\forall \alpha \in \mathcal{V}_2,\ \rho(y; \alpha) = o(|y|^p)$

Consider the case Hopf case: $\mathbf{L} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, $\omega > 0$. In the basis $(\zeta, \bar{\zeta}), \zeta = (1, -i)$: $\mathbf{L} = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$. Write $x = y + \Phi_\alpha(y)$, the change of variable with $y = A\zeta + \overline{A\zeta}$

1. Prove that $\mathbf{N}_\alpha(A\zeta + \overline{A\zeta}) = AQ_\alpha(|A|^2)\zeta + \overline{AQ_\alpha}(|A|^2)\bar{\zeta}$ where Q_α is a polynomials.
2. Write the vector field at order 3 in A . Do you recognize something?