ENS - Master MVA / Paris 6 - Master Maths-Bio

Tutorial 3

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Exercice

Bogdanov-Takens Normal form

Assume that the jacobian of a DS at $u = 0$ is given by its Jordan normal form $L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ where $u = (A, B) \in \mathbb{R}^2$.

1. Show that the polynomial N of the normal form is [N(u)=(AP(A),BP(A)+Q(A))] where P, Q are real-valued polynomials, satisfying $P(0) = Q(0) = Q'(0) = 0$.
2. Using a change of variables, show that the DS can be modified into

$$\begin{cases} dA/dt = \tilde{B} \\ d\tilde{B}/dt = \tilde{B}P_1(A) + Q_1(A) + \tilde{\rho}_1(A, \tilde{B}) \end{cases}$$

Exercice

Wilson-Cowan

This is the main mechanism to produce cortical oscillations with two interacting populations (PING mechanism). Consider two populations E/I with the following dynamics

$$\begin{cases} \dot{E} & = & -E + S(J_{EE}E + J_{EI}I + \theta_E) \\ \dot{I} & = & -I + S(J_{IE}E + J_{II}I + \theta_I) \end{cases}$$

where S is the sigmoid function

$$S(x) = \frac{1}{1+e^{-x}}.$$

1. Write the equation for the equilibrium. Write the jacobian L of the system (Hint: $S' = S(1-S)$).

2. Write the **linear** conditions for the Hopf bifurcation and find a way to compute the Hopf bifurcation curves in the plane (θ_E, θ_I).

3. Can we do the same for the Saddle-Node bifurcation curve?

Exercice

Delayed inhibition

This is the main mechanism to produce cortical oscillations with inhibitory neurons (ING mechanism). Consider one population of such neurons I with the following dynamics

$$\tau \dot{I}(t) \quad = \quad -I(t) + JS(\sigma I(t-D) + \theta) \in \mathbb{R}$$

where S is the sigmoid function (see above), $\sigma \geq 0$ is the nonlinear gain and $J < 0$. **On se restreint au cas $\tau = 1$.**

1. Show that there is a unique negative stationary state I_σ^{eq} that is monotonic in σ.
2. Write the linear equation around I_σ^{eq} and look for perturbation $e^{\lambda t}U$. Find an equation for λ.
3. Write the solutions of this equations using the different solutions $W_k(z)$ of the equation $we^w = z$. This function is called the Lambert function. You have computed the spectrum.
4. We change of method. Give a necessary condition on $\sigma J S'$ in order to have a Hopf bifurcation. In this case, show that the critical delay is $D = \frac{1}{\sqrt{J^2-1}}\left(\pi - \arccos\left(\frac{1}{|J|}\right)\right)$.
5. Show that $\sigma \to \sigma S'(\sigma I_\sigma^{eq} + \theta)$ is increasing. Conclude on the existence of a Hopf bifurcation when increasing the nonlinear gain.