ENS - Master MVA / Paris 6 - Master Maths-Bio

Tutorial 4

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Exercise

Around the neural field equation (NFE)

We consider a NFE on a compact domain $\Omega\subset\mathbb{R}^p$ with a sigmoid S nonlinearity:

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_\Omega w(x,y)S(V(y,t))dy.$$

1. We assume that $w\in C^0(\Omega^2,\mathbb{R})$. Prove existence / uniqueness of the solution in the space $\mathcal{C}=\mathcal{C}(\Omega,\mathbb{R})$. *Hint:* show that it is globally Lipschitz. (This implies that the solution is defined globally, ie on $\mathbb{R}$.)
2. Show that the nonlinearity is C^1.

   We focus on the case $w(x,y) = w_0 + w_1\cos(x-y)$ on $\Omega = (-\pi,\pi)$

3. Write the equations satisfied by the equilibrium. Are they finite dimensional?
4. Write a simplified set of equations for the dynamics. Hint: decompose the space with the range of the integral convolutional operator.
5. Consider a stationary state. Can you study its stability despite the fact that the equations are infinite dimensional? (Can you find a case where you can...?)

> 1/ This is a consequence of Cauchy Lip. theorem. We check that the right-hand side $F(V)$ is globally Lipschitz by noting that $S_m := \sup S'(x) < \infty$. Hence

> $\|F(V_1) - F(V_2)\|_\infty \leq (1 + S_m\|w\|_\infty)\|V_1 - V_2\|_\infty$. It implies that the NFE has a solution globally defined in time. This is a classical consequence of the Cauchy-Lipschitz theorem but not of the finite time explosion (because we are in infinite dimension). 2/ We focus on $G(V) = W \cdot S(V)$. From Taylor with integral reminder, we have $S(V+U) - S(V) - US'(V) = \int_0^1 (1-t)S^{(2)}(V+tU)U^2 dt$. This shows that $W \cdot S(V+U) - W \cdot S(V) - W \cdot US'(V) = \int_0^1 (1-t)WS^{(2)}(V+tU)U^2 dt$. One can check that the linear operator $\mathbf{L} = U \to W \cdot US'(V)$ is continuous on $\mathcal{C}$ and is thus the candidate for the differential of G at V. It remains to check that the integral term is $o(U)$. The reminder is $\int_0^1 (1-t)W \cdot S^{(2)}(V+tU)U^2 dt$ and its norm is bounded by $\sup_x S^{(2)}(x)\|w\|_\infty\|U\|^2 = o(U)$. This shows that G is differentiable.
>
> 3/ Stationary solutions are $V(x) = \int_\Omega w(x-y)S(V(y))dy = w_0 \int_\Omega S(V) + w_1 \sin(x) \int_\Omega \sin S(V) + w_1 \cos(x) \int_\Omega \cos S(V)$. Hence, there are v_0, v_1, v_2 so that $V(x) = v_0 + v_1 \cos + v_2 \sin$. We can plug this expression in the integral equation and project on $1, \cos, \sin$. It gives 3 equations in v_0, v_1, v_2. The problem is 3d. 4) Same technics as 3). 5) Let us imagine that $V^{eq} = 0$ is a stationary solution. In the general case, one has to adapt a bit the technics but it is essentially the same idea. Then, the stability is linked to the spectrum of the linearized operator $LU = -U + W \cdot (S'(0)U)$. Hence, we focus on the second term. This operator is of rank 3. In fact, in the orthogonal of $Vect(1, cos, sin)$, it is zero. One can thus restrict the study on $Vect(1, cos, sin)$ where we find that the spectrum is $\{2\pi w_0, \pi w_1\}$. All in all, the spectrum of the linearized operator is $\{-1, -1 + 2\pi w_0, -1 + \pi w_1\}$. The stability analysis is then straightforward.

Exercise

Around the neural field equation of Amari type

We consider a neural field equation on the real line

$$\frac{d}{dt}V(x,t) = -V(x,t) + \int_{\mathbb{R}} w(x-y)S(V(y,t))dy + h$$

in the case where $S(v) = \mathbf{1}_{v>0}$ is the Heaviside function and $h \in \mathbb{R}$.

The **connectivity kernel** $w \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a real **even** function which is integrable on $\mathbb{R}$. We define $W(x) = \int_0^x w$ and $W_\infty := \lim_{x\to\infty} W(x)$.

We further define $R(V) = \{x,\ V(x) > 0\}$. An equilibrium V^{eq} is said **localized** if

$R(V^{eq}) = (a_1, a_2)$ with $a_i \in \mathbb{R}$. In this case, we can always assume $a_1 = 0$ by translation invariance.

1. An equilibrium V^{eq} such that $R(V^{eq}) = \emptyset$ exists if and only if $h \leq 0$.
2. An equilibrium V^{eq} such that $R(V^{eq}) = \mathbb{R}$ exists if and only if $2W_\infty > -h$.
3. We here assume the following behavior of W on $\mathbb{R}^+$. W is strictly increasing towards its maximal value W_M and is then stricly decreasing and converging to $W_\infty < 0$. Show that an equilibrium V^{eq} such that $R(V^{eq}) = (0, a)$ with $a > 0$ exists if and only if $h < 0$ and $W(a) + h = 0$.
4. Find solutions with a periodic support, namely $R(V^{weq}) = \cup_{n=-\infty}^{\infty}[-b + nL, b + nL]$ under the restriction $2b < L$. Find an equation satisfied by b.
5. Find traveling fronts $V(x, t) = U(x - ct)$ where the speed c and the waveform U have to be determined. One can introduce traveling wave coordinate $\xi = x - ct$ and assume $c > 0$.
6. **Interface dynamics**. We assume that a solution is such that $R(V(x, t)) = (-a(t), a(t))$ and that $0 \leq V_0(x) \leq 1$. We assume that V_0 is even. Find an equation satisfied by a in the case $h = 0$.
7. In the case of a non-convolutional kernel, $w(x, y) = e^{-|x-y|}(1 + a\cos(y)), 0 < a < 1$, find the number of stationary solutions as function of their width.

> This behavior is called **snaking** of stationary solutions.
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> 1/ If such solution exists then $V(x) = h$ which requires $h \leq 0$. On the contrary, if $h \leq 0$, then $V(x) = h$ is a stationary solution of the NFE.
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> 2/ If there is such solution then it satisfies $V(x) = \int_{\mathbb{R}} w(x - y)dy + h = 2W_\infty + h > 0$. On the contrary, if $2W_\infty + h > 0$, then $V(x) = 2W_\infty + h$ is such solution.
>
> 3/ A **localised** solution is such that $V(x) = \int_0^a w(x - y)dy + h = W(x) - W(x - a) + h$. This solution is continuous, hence it satisfies $V(0) = V(a) = 0$ which implies $W(a) + h = 0$. Finally, $V \to h$ when $x \to \infty$ which implies $h \leq 0$. On the contrary, when $W(a) = -h$ holds and $h \leq 0$, one finds $V(a) = V(0) = 0$. Moreover, such V is C^1. Using the hypothesis on the shape of W, one can prove that $V(x) = W(x) - W(x - a) + h$ is positive on the interval $(0, a)$ and negative elsewhere, provided that $h \leq 0$.
>
> 4/ These solutions take the form $U(x) = \sum_{n\in\mathbb{Z}} \int_{-b+nL}^{b+nL} w(x - y)\mathrm{d}y + h = \sum_{n\in\mathbb{Z}}(W(x + b + nL) - W(x - b + nL)) + h$. The threshold condition reads $U(\pm b + nL) = 0$. This gives $h + \sum_{n\in\mathbb{Z}}(W(2b + nL) - W(nL)) = 0$.

> 5/ the solution satisfies $U_f(\xi) = \mathrm{e}^{\xi/c}\left[\kappa - \frac{1}{c}\int_0^\xi \mathrm{e}^{-y/c}\left(W_\infty - W(y)\right)\mathrm{d}y\right]$. Assuming $c > 0$ and requiring boundedness implies $\kappa = \frac{1}{c}\int_0^\infty \mathrm{e}^{-y/c}\left(W_\infty - W(y)\right)\mathrm{d}y$. The traveling wave is thus of the form $U_f(\xi) = \frac{1}{c}\int_0^\infty \mathrm{e}^{-y/c}\left(W_\infty - W(y+\xi)\right)\mathrm{d}y$.
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> 6/ By definition $V(\pm a(t), t) = 0$ hence $\pm x(t)a'(t) + \partial_t V(\pm a(t), t) = 0$ where we defined $\pm\alpha(t) = \partial_x u(\pm a(t), t)$. One gets $a'(t) = -\frac{1}{\alpha(t)}[W(2a(t)) - \kappa]$. This equation is not well defined for $\alpha(t) = 0$. We now get an expression for α. We define $z(x,t) = \partial_x V(x,t)$ and find $\partial_t z(x,t) = -z(x,t) + w(x + a(t)) - w(x - a(t))$ which allows to find $\alpha(t) = u_0'(a(t))\mathrm{e}^{-t} + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s[w(a(t) + a(s)) - w(a(t) - a(s))]\mathrm{d}s$. This gives us a closed system describing the evolution of a, α with the initial conditions $a(0) = l, \alpha(0) = u_0'(l)$ as long as $\alpha < 0$.
>
> 7) The width a of the solution satisfies $h + \int_0^a w(a,y)dy = 0$. Writing $W(x) = \int_0^x w(x,y)dy$, one finds for $x > 0$, $W(x) = 1 - e^{-x}\frac{a}{2} - e^{-x} + a\cos(x)/2 + a\sin(x)/2$. Hence, there are infinitely many as solution for $h \approx 1$.

Exercise

Normal form and center manifold

We consider $\frac{du}{dt} = \mathbf{A}u + \mathbf{R}(u,\mu)$. Let us write the Taylor expansion of $\mathbf{R}$ for a given p:

$$\mathbf{R}(u) = \sum_{2\le q+l\le p} \mathbf{R}_{ql}[u^{(q)}, \mu^{(l)}] + o\left(\|u\|^p\right),\ \mathbf{R}_{01} = 0$$

with $\mathbf{R}_{ql} = \frac{1}{q!l!}\frac{\partial\mathbf{R}}{\partial^q u \partial^l \mu}$, $u^{(q)} \equiv (u, \cdots, u) \in \mathcal{Z}_h^q$ and $\mu^{(l)} \equiv (\mu, \cdots, \mu) \in (\mathbb{R}^{m_{par}})^l$.

1. Assume that there is a center manifold Ψ and that we perform a Normal form simplification *on* the center manifold with change of variable $u_c = v_0 + \Phi_\mu(v_0)$. Write the nonlinear mapping $v_0 \to u$. Write the equation satisfied by the combined change of variables (Hint: use the same technics as for the center manifold function Ψ).
2. (Difficult) Assume that we have a Hopf bifurcation. We write $v_0(t) = A(t)\zeta + \overline{A(t)\zeta}$ with $A \in \mathbb{C}$ where v_0 is the coordinate on the center manifold after the normal form transform. Assume that the normal form reads $\dot{A} = A\left(a\mu + b|A|^2\right) + O((|\mu| + |A|^2)^2)$. Finally write $u = v_0 + \tilde{\Psi}(v_0,\mu)$ and $\tilde{\Psi}(v_0,\mu) = \sum_{p,q,r}\Psi_{p,q,r}A^p\bar{A}^q\mu^r + \cdots$. Show that $a = \langle \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \Psi_{001}), \zeta^*\rangle$ and

$b = \langle 2\mathbf{R}_{20}(\zeta, \Psi_{110}) + 2\mathbf{R}_{20}(\bar{\zeta}, \Psi_{110}) + \mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta}), \zeta^* \rangle$. For this, find the first terms $\Psi_{001}, \ldots$ by identifying the monomials of $\tilde{\Psi}$ in the equation derived in 1).

3. We consider the **Ring Model of orientation tuning** $\dot{V} = -V + J \star S_0(V)$ on the circle, ie $J \star S_0(V)(\theta) = \int_{-\pi}^{\pi} J(\theta - \theta') S_0(V(\theta')) d\theta'$, V is periodic and S_0 is a smooth bounded function such that $S_0(0) = 0$. Assume further that w is **even**.
   1. Show that the vector field is equivariant w.r.t $T_t \cdot V(\theta) = V(\theta - t)$ and $R \cdot V(\theta) = V(-\theta)$. What is the group generated by T_t, R?
   2. ($\star\star$) Assume that there a Pitchfork bifurcation at $\mu = \mu_0$. Compute the normal form as function of the parameters of the model. For this, one has to adapt the previous question 2).

ATTENTION ICI NOTATIONS

> 1. we build a reduced equation for $u_c \in \mathcal{X}_c$ with the center manifold correction Ψ:
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> $$u = u_c + \Psi(u_c, \mu), \quad \Psi(u_c, \mu) \in \mathcal{Z}_h.$$
>
> This reduced equation is
>
> $$\frac{du_c}{dt} = \mathbf{A} u_c + P_c \mathbf{R}\left(u_c + \Psi(u_c, \mu), \mu\right).$$
>
> Then, we apply a change of variable to u_c
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> $$u_c = v_0 + \Phi_\mu(v_0), \quad v_0 \in \mathcal{X}_c$$
>
> to bring the reduced equation to a normal form given by:
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> $$\frac{dv_0}{dt} = \mathbf{A}|_{\mathcal{X}_c} v_0 + \mathbf{N}_\mu(v_0) + \rho(v_0, \mu),$$
>
> where $\mathbf{N}_\mu$ is a polynomial of some degree p such that $\mathbf{N}_0(0) = 0,\ D_v \mathbf{N}_0(0) = 0$ and $\rho(v_0, \mu) = o(\|v_0\|^p)$. We write
>
> $$u = v_0 + \tilde{\Psi}(v_0, \mu), \quad \tilde{\Psi}(v_0, \mu) \equiv \Phi_\mu(v_0) + \Psi(v_0 + \Phi_\mu(v_0), \mu) \in \mathcal{Z}$$
>
> The nonlinear function $\tilde{\Psi}$ is solution of the next equations:
>
> $$(NF): \begin{cases} D_{v_0} \tilde{\Psi}(v_0, \mu) \mathbf{A}|_{\mathcal{X}_c} v_0 - \mathbf{A} \tilde{\Psi}(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}(v_0) \\ \mathbf{Q}(v_0) \equiv \Pi_p \left[\mathbf{R}(v_0 + \tilde{\Psi}(v_0, \mu), \mu) - D_{v_0} \tilde{\Psi}(v_0, \mu) \mathbf{N}_\mu(v_0) \right] \end{cases}$$
>
> where Π_p is the operator which takes the first $p + 1$ terms in the Taylor expansion in

the variable v_0.

2. Under these assumptions, we have that $\Sigma_0 = \{\pm i\omega\}$ and that the associated center subspace E_0 is two-dimensional spanned by the eigenvectors $\zeta, \bar{\zeta}$ associated with iω and –iω, respectively. We set $\tilde{\Psi} \equiv \sum_{p,q,r} \Psi_{p,q,r} A^p \bar{A}^q \mu^r$. By identifying in (NF) the terms of order $O(\mu)$, $O(A^2)$, and $O(A\bar{A})$, we obtain

$$-\mathbf{L}\Psi_{2,0,0} = \mathbf{R}_{0,1}$$

$$(2i\omega - \mathbf{L})\Psi_{2,0,0} = \mathbf{R}_{2,0}(\zeta, \zeta)$$

$$-\mathbf{L}\Psi_{1,1,0} = 2\mathbf{R}_{2,0}(\zeta, \bar{\zeta})$$

Here the operators $\mathbf{L}$ and $2i\omega - \mathbf{L}$ are invertible so that the equation above are uniquely determined. Next we identify terms of order $O(\mu A)$ and $O(A^2\bar{A})$ and find

$$(i\omega - \mathbf{L})\Psi_{1,0,1} = -a\zeta + \mathbf{R}_{1,1}(\zeta) + 2\mathbf{R}_{2,0}(\zeta, \Psi_{0,0,1})$$

$$(i\omega - \mathbf{L})\Psi_{2,1,0} = -b\zeta + \mathbf{R}_{2,0}(\zeta, \Psi_{1,1,0}) + 2\mathbf{R}_{2,0}(\bar{\zeta}, \Psi_{2,0,0}) + 3\mathbf{R}_{3,0}(\zeta, \zeta, \bar{\zeta})$$

Since iω is a simple isolated eigenvalue of $\mathbf{L}$, the range of $(i\omega - \mathbf{L})$ is of codimension 1, so that we can solve these equations, and determine Ψ1,0,1 and Ψ2,0,0, provided the right hand sides satisfy one solvability condition. It is this solvability condition which allows to compute the coefficients a and b. In the case when $\mathbf{L}$ has an adjoint $\mathbf{L}^*$, acting in the dual space X^*, then the solvability condition is that the right hand sides are orthogonal to the kernel of the adjoint $(-i\omega - \mathbf{L}^*)$ of $(i\omega - \mathbf{L})$. The kernel of $(-i\omega - \mathbf{L}^*)$ is one-dimensional, just as the kernel of $(i\omega - \mathbf{L})$, spanned by $\zeta^* \in \mathcal{X}^*$ that we choose such that $\langle \zeta, \zeta^* \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^*. For example $a = \langle \mathbf{R}_{1,1}(\zeta) + 2\mathbf{R}_{2,0}(\zeta, \Psi_{0,0,1}), \zeta^* \rangle$.

3. By symmetry arguments, the bifurcation is of "pitchfork" type and the central part is given by $E_0 = Vect(\cos(2\cdot), \sin(2\cdot)) = \ker \mathbf{L}_{\mu_0}$ where $\mathbf{L}_\mu = -Id + \mu S'(\mu v_0^f(\mu))J$. We use complex coordinates $U_0 = A\zeta + c.c$ with $\zeta = e^{2i\theta}$ and $A \in C$, the normal form is given by:

$$\dot{A} = A\left(\frac{\sigma - \sigma_0}{\sigma_0} + \chi_3 |A|^2\right) + h.o.t.$$

We write $V(\theta, t) = v_0^f + \mathbf{U}_0 + \Psi(\mathbf{U}_0, \mu)$. The Taylor expansion reads

$$\Psi(\mathbf{U}_0, \mu) = \Psi_{20}A^2 + \bar{\Psi}_{20}\bar{A}^2 + \Psi_{11}A\bar{A} + \Psi_{30}A^3 + \bar{\Psi}_{30}\bar{A}^3 + \Psi_{21}A^2\bar{A} + \bar{\Psi}_{21}A\bar{A}^2$$

We write $\mathbf{R}(\mathbf{U},\mu) = -v_0^f - \mathbf{U} + \mathbf{J} \star S(\mu\mathbf{U} + \mu v_0^f) - \mathbf{L}_{\mu_0}$. We note that $D_{v_0}\tilde{\Psi}(v_0,\mu)\mathbf{A}|_{\mathcal{X}_c} v_0 = 0$. We look at the coefficient $A|A|^2$ in (NF), it satisfies

$$b\zeta - \mathbf{L}_{\mu_0}\Psi_{21} = 2\mathbf{R}_{20}(\zeta,\Psi_{11}) + 2\mathbf{R}_{20}(\bar{\zeta},\Psi_{20}) + 3\mathbf{R}_{30}(\bar{\zeta},\zeta,\zeta).$$

The coefficient A^2 gives

$$-\mathbf{L}_{\mu_0}\Psi_{20} = \mathbf{R}_{20}(\zeta,\zeta)$$

The coefficient $A\bar{A}$ gives

$$-\mathbf{L}_{\mu_0}\Psi_{11} = \mathbf{R}_{20}(\zeta,\bar{\zeta})$$

The coefficient A^3 gives

$$-\mathbf{L}_{\mu_0}\Psi_{30} = \mathbf{R}_{20}(\zeta,\Psi_{20}) + \mathbf{R}_3(\bar{\zeta},\zeta,\zeta).$$

It remains to solve the above convolution equations. We write $e_n(\theta) = e^{2in\theta}$ and $J \star e_n = J_n e_n$. We find $\mathbf{R}_2(\zeta,\zeta) = \frac{1}{2}\mu_0^2 S^{(2)}(\mu_0 v_0^f) J \star \zeta^2$ hence $\Psi_{20} = \alpha_{20}\zeta + \beta_{20}\bar{\zeta} + \frac{\mu_0^2 S^{(2)}(\mu_0 v_0^f) J_2}{2(-1+\mu_0 S^{(1)}(\mu_0 v_0^f) J_2} = \alpha_{20}\zeta + \beta_{20}\bar{\zeta} + \frac{\mu_0^2 S^{(2)}(\mu_0 v_0^f) J_2}{2(1-J_2/J_1)}$. The final result is

$$b = \mu_0^3 J_1 \left[\frac{S^{(3)}(\mu_0 v_0^f)}{3} + \mu_0 S^{(2)}(\mu_0 v_0^f) \left(\frac{J_0}{1 - J_0/J_1} + \frac{J_2}{2(1 - J_2/J_1)} \right) \right]$$