Paris - Master Mathbio / Master MVA
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2021-2022

Exam - Mathematical Methods in Neurosciences
January 6th, 2022

Part 1 - Lectures questions

1. What can you say about the columnar organization of the visual area V1?
2. What can you say about synaptic plasticity?
3. Can you give examples in the neurosciences field where the use of delay differential equations can be interesting?

Consider the equation $\frac{d}{dt}x = \mu \cdot x - x^3 + R(x) \in \mathbb{R}$ with $R(x) = o(x^3)$ and $\mu \in \mathbb{R}$.

4. Study the truncated system (for $R = 0$) as function of $\mu \in \mathbb{R}$ when x is close to zero. In particular, gives the equilibria, their stability and the flow.
5. When $R \neq 0$, do the results of the previous question persist?
6. Give the name of this bifurcation.

Part 2 - Spiking times

Constant spike Rates

We assume in this part that an individual neuron spikes at constant rate $\mathbf{r}$. We denote by $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_\ell \leq \cdots$ the sequence of spiking times. The number of spikes between times 0 and t is denoted $K_t := \sum_{\ell \geq 1} \mathbb{1}_{\{\tau_\ell \leq t\}}$.

7. Give the law of the first spiking time τ_1.
8. Give the Laplace transform of τ_1 defined by $\Psi(z) = \mathbb{E}[\exp(-z\tau_1)]$.
9. Is the stochastic process $(K_t)_{t \geq 0}$ a Markov process? Give its infinitesimal generator L.

Non constant Rates

In this question, we no more assume that the spike rate is constant but it is a deterministic function $r(t)$ of time t. We still denote by $(\tau_\ell)_{\ell \geq 1}$ the sequence of spiking times and by K_t the *counter* of spikes.

10. Give an efficient algorithm to simulate τ_1.
11. Recall without proof the tail distribution g of τ_1, $g(\theta) := \mathbb{P}[\tau_1 > \theta]$.
12. Give the infinitesimal generator L^t of the process $(K_t)_{t \geq 0}$

Random Rates depending on the membrane potential

We now assume that the spike rates in no more a deterministic *external* function but depends on the value of the membrane potential V_t of the neuron. We assume that **between the spikes**, the membrane potential evolves according to the differential equation

$$\frac{dV_t}{dt} = b(V_t). \tag{1}$$

The neuron spikes at rate $\lambda(V_t)$, that is

$$\lim_{\eta \to 0} \frac{1}{\eta} \mathbb{P}(\text{neuron spikes} \in [t, t+\eta] \,|\, V_t) = \lambda(V_t).$$

When a spike occurs, the membrane potential is reset to V^r. We still denote by $(\tau_\ell)_{\ell \geq 1}$ the sequence of spiking times and by K_t the *counter* of spikes.

13. Assume that for any sequence of times $s_1, \cdots, s_k$, we are able to find directly the value of the solution of (1), say $\tilde{V}_{s_1}, \cdots, \tilde{V}_{s_k}$. Give an efficient algorithm of simulation of the sequence $(\tau_\ell)_{\ell \geq 1}$.
14. Give the infinitesimal generator of (V_t, K_t).
15. Is the counting process $(K_t)_{t \geq 0}$ a Markov process? Is the membrane potential V_t a Markov process? For both processes, if it is Markov, give the infinitesimal generator.
16. **Example 1:** We now assume that $V^r = 0$, $b(v) = v$, $\lambda(v) \equiv 1$ and V_0 is uniformly distributed on $[V^r, V^r + 1]$. Give the distribution of V_t. Give the asymptotic distribution of (V_t) as t goes to infinity.
17. **Example 2:** Same questions if $\lambda(v) = v$, the other parameters and functions being similar as in Example 1. In addition, compute the spike rate r_t of the neuron.

Random Rates depending on the membrane potential and the time spent since the last spike

In this part, we assume that the spiking rate also depends on $t - \tau_{\alpha(t)}$, where $\alpha(t) = \sup\{\tau_k \text{ such that } \tau_k < t\}$. We denote the spiking rate $\lambda(V_t, t - \tau_{\alpha(t)})$.

18. Give a Markov process associated to this setting. Give its infinitesimal generator.

Part 3 - Feed-forward networks

This part concerns the study of feed-forward chains that are used to model synchronization in spike rate based neural networks. We are interested in understanding how the dynamical singularities of isolated populations are modified when the populations are coupled.

We consider the system in $\mathbb{R}^m \times \mathbb{R}^m$ given by

$$\begin{aligned} \dot{x} &= f(x, 0) \\ \dot{y} &= f(y, x) \end{aligned} \tag{2}$$

where $f(0,0) = 0$. We further assume that the linearisation at the origin of the vector field is $J = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$.

19. Show that the flow can be written $\Phi^t(x(0), y(0)) = (\phi^t(x(0), 0), \phi^t(y(0), x(0)))$.

We recall the notation of the center part $\Sigma_c(a) = \{\Re = 0\} \cap \Sigma(a)$ of the spectrum of a, defined in J, and its associated generalized eigenspace $\mathbf{E}_c(a) = \oplus_{\lambda\in\Sigma_c(a)} \oplus_{p\geq 1} \ker(\lambda I - a)^p$.

20. We assume that the center subspace $\mathbf{E}_c(a)$ of a is of dimension $n \leq m$. Show that the first equation of (2) has a locally invariant manifold $\mathcal{V}_c$.

21. Show that (2) has a locally invariant manifold $\mathcal{W}_c$.

22. Let us define the projection $\Pi(x, y) = (x, 0)$. Show that $\Pi^{-1}(\mathcal{V}_c \times \{0\})$ is flow invariant.

23. Show that one can choose a $2n$-dimensional center manifold $\mathcal{W}_c \subset \mathbb{R}^m \times \mathbb{R}^m$ for (2) so that $\Pi(\mathcal{W}_c) = \mathcal{V}_c \times \{0\}$. (One can reduce $\mathcal{V}_c$.)

We admit that there is an invertible mapping $P : \mathcal{V}_c \times \mathcal{V}_c \to \mathcal{W}_c$ such that $P : (z_2, z_3) = (z_2, \rho(z_2, z_3))$, $P(z_2, 0) = (z_2, 0)$ and $P(0, z_3) = (0, z_3)$.

24. Compute the flow of (2) on $\mathcal{V}_c \times \mathcal{V}_c$ as function of ϕ^t and the inverse of P.

25. Deduce that the dynamics on the center manifold $\mathcal{W}_c$ of (2) can be written on $\mathcal{V}_c \times \mathcal{V}_c$ as

$$\begin{aligned} \dot{z}_2 &= g(z_2, 0) \\ \dot{z}_3 &= g(z_3, z_2) \end{aligned} \tag{3}$$

for some function g and coordinates $z_2, z_3 \in \mathcal{V}_c$. (One could use question 19)

Study of the normal form.

26. Assume that the center subspace $\mathbf{E}_c$ is two-dimensional, and the linearization of the internal dynamics a has a pair of purely imaginary eigenvalues. We rescale time so these are equal to $\pm i$. Does the normal form has the same structure as (4)?

27. Show that any change of variable $(z_2, z_3) \to (Q(z_2), Q(z_3))$ which leaves $(0, 0)$ invariant does change the structure of (3). Deduce that we can assume $g(z_2, 0)$ being in normal form. What is the expression of this normal form?

28. We assume that for any value of k, there exists a polynomial change of coordinates which can transform (3) into the system in $\mathbb{C} \times \mathbb{C}$

$$\begin{aligned} \dot{z}_2 &= p_k(z_2, \overline{z_2}) + h.o.t. \\ \dot{z}_3 &= p_k(z_3, \overline{z_3}) + q_k(z_3, \overline{z_3}, z_2, \overline{z_2}) + h.o.t. \end{aligned} \tag{4}$$

where p_k and q_k are polynomials of order k, $q_k(z_3, \overline{z_3}, 0, 0) = 0$, h.o.t. indicates terms of degree at least k + 1 and the truncated system is equivariant under the action of $\mathbf{S}^1$, $\theta \cdot (z_2, z_3) = (e^{i\theta} z_2, e^{i\theta} z_3)$. What is the general expression for p_k, q_k?