Paris 6 Master Mathbio / ENS / Master MVA 2015-2016
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Exam - Mathematical Methods in Neuroscience
January 7th, 2016

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} \equiv (\mathcal{F}_t)_{t\geq 0}$. It is assumed that the filtration satisfies the so-called usual conditions: for every t in $\mathbb{R}^+$, $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{\theta > t}\mathcal{F}_\theta$ and $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets in $\mathcal{F}$.

1 Questions de cours

1.1 Poisson Processes

We consider a counting process $(N_t)_{t\geq 0}$, adapted to the filtration $\mathbb{F}$. We assume that for all $s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$ and that the law of $N_t - N_s$ only depends on $t - s$ (we say that N is a counting process with independent increments and stationary increments).

1. *Prove that N is a Poisson process.*
   Reminder

$$\text{(Stirling)} \quad n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad \lim_n \left(1 + \frac{u}{n}\right)^n = \exp(u).$$

Proof. **Step 1**
For any $n \geq 1$, we write $p_n = \mathbb{P}\left(N((k+1)/n) - N(k/n) \geq 1\right).$

$$\begin{aligned}\mathbb{P}(N(1) = 0) &= (1 - p_n)^n \\ \lambda &= -\log(\mathbb{P}(N(1) = 0)) \\ &= -n\log(1 - p_n) \\ &= \lim_n np_n. \qquad \Rightarrow \qquad \mathbb{P}(N(1) = 0) = \exp(-\lambda)\end{aligned}$$

Step 2 Let denote $q_n = \mathbb{P}(N(1/n) \geq 2)$. Denote Γ_n the number of intervals $[k/n, (k+1)/n]$ containing at least 2 arrivals.

- $\Gamma_n(\omega) \to 0$ for almost all ω (the time arrival are different).
- $\Gamma_n \leq N(1)$
- We have $\mathbb{E}(N(1)) < \infty$(admitted)
- So, we conclude $\mathbb{E}(\Gamma_n) \to_n 0$ (Fubini), that is nq_n tends to 0.

Step 3
We deduce that $\lim_n n\mathbb{P}(N(1/n)=1)=\lim_n n\mathbb{P}(N(1/n)\geq 1)=\lambda$. So, asymptotically, the process $N(s)$To finish the proof, we consider the random walk S_n:

$$S_0=0, \qquad S_{k+1}=S_k+X_{k+1},$$

where $X_1,\cdots,X_k,\cdots$ are *i.i.d.* random variables

$$\mathbb{P}(X_k=1)=\frac{\lambda}{n}=1-\mathbb{P}(X_k=0).$$

Let $\lambda>0$ and $t_1<t_2<\cdots<t_\ell$ and consider sequences, $n_1(n),\cdots,n_\ell(n)$ such that

$$\forall i\in 1,\cdots,\ell, \quad \lim_n \frac{n_i(n)}{n}=t_i,$$

We have

$$(S_{n_i(n)}-S_{n_{i-1}(n)}, 1\leq i\leq \ell) \xrightarrow[n\to\infty]{\mathcal{L}} (Y_1,Y_2,\cdots,Y_\ell)$$

where Y_i are independent r.v. with Poisson laws of parameters $\lambda(t_i-t_{i-1})$.

$$\mathbb{P}(S_{n_1(n)}=k)=\frac{(t_1n)!}{k!(t_1n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{t_1n-k}$$

$$\text{(Stirling)} \quad n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad \lim_n\left(1+\frac{u}{n}\right)^n=\exp(u).$$

$$\begin{aligned}\mathbb{P}(S_{n_1(n)}=k)&\approx\frac{\sqrt{2\pi t_1 n}}{\sqrt{2\pi(t_1n-k)}}\left(\frac{t_1n}{e}\right)^{t_1n}\left(\frac{e}{t_1n-k}\right)^{t_1n-k}\frac{1}{n^k}\frac{\lambda^k}{k!}\exp(-\lambda t_1)\\&\approx\exp(-\lambda t_1)\frac{(\lambda t_1)^k}{k!}e^{-k}\left(1+\frac{k}{t_1n-k}\right)^{t_1n}\approx\exp(-\lambda t_1)\frac{(\lambda t_1)^k}{k!}.\end{aligned}$$

□

1.2 Deterministic systems

We consider the scalar system

$$\dot{u}(t)=f(u(t),\mu).$$

We assume that f is C^k with $k\geq 3$ in a neighborhood of $(0,0)$ and that $f(0,0)=0$, $\frac{\partial f}{\partial u}(0,0)=0$.

We further assume that f is odd in u: $f(u,\mu)=-f(-u,\mu)$ and that

$$\frac{\partial^2 f}{\partial\mu\partial u}(0,0):=1, \ \frac{\partial^3 f}{\partial u^3}(0,0):=-6$$

2. Analyse the system when f is truncated at order $o(u|\mu|+u^3)$. Draw the bifurcation diagram as function of μ. What is the name of this bifurcation?

Proof. The normal form reads $\dot{x} = x(\mu - x^2)$. This is the Pitchfork bifurcation. $x = 0$ is stable (resp. unstable) for $\mu < 0$ (resp. for $\mu > 0$). When $\mu > 0$, there are two nontrivial (stable) equilibria $x_\pm = \pm\sqrt{\mu}$. □

3a. We admit that f can be written $f(u, \mu) = uh(u^2, \mu)$ with h $C^{(k-1)/2}$ in a neighbourhood of $(0, 0)$. Show that there is a unique equilibrium point for $\mu < 0$, discuss its stability.

3b. For $\mu > 0$, the system possesses the trivial equilibrium $u = 0$ and two nontrivial equilibria $u_\pm(\epsilon), \epsilon = |\mu|^{1/2}$, which are symmetric, $u_i(\epsilon) = -u_+(\epsilon)$. The map $\epsilon \to u_\pm(\epsilon)$ is of class C^{k-3} in a neighbourhood of 0, and $u_\pm(\epsilon) = O(\epsilon)$. Give their stability.

Proof. 3a-b) We apply the implicit function theorem to the equation $h(u^2, \mu) = 0$ which shows that it has a unique solution $\mu = g(u^2)$ with $g(0) = 0$ and g of class $C^{(k-1)/2}$ in a neighborhood of 0. The Taylor expansion of g is given by $\mu = -u^2 + O(u^4)$. We then conclude that there is a curve of nontrivial equilibria in the (μ, u)-plane that has a second order tangency at $(0, 0)$ to the parabola $\mu = -u^2$ found for the truncated equation, and which is symmetric with respect to the μ-axis. Again, this shows that the truncated equation and the full equation have the number of equilibria in a neighborhood of the origin, which are $o(\sqrt{|\mu|})$ - close to each other. As for the dynamics, it is here again easy to study by looking at the sign of $f(u, \mu)$. □

2 Invariant measures

We consider the process $(X, Y) \in \mathbb{R}^2$, solution of the SDE

$$X_t = x + \int_0^t \alpha X_s ds + \int_0^t \sigma X_s dW_s$$
$$Y_t = \int_0^t X_s ds.$$

4. (course) Give an explicit expression for X.

Proof.

$$X_t = x \exp\left((\alpha - \frac{\sigma^2}{2})t + \sigma W_t\right)$$

□

5. Study the deterministic system when $\sigma = 0$. Plot the bifurcation diagram.

We recall the Ito's formula:

$$F(X_t, Y_t) = F(X_0, Y_0) + \int_0^t \frac{\partial}{\partial x} F(X_s, Y_s) dX_s + \int_0^t \frac{\partial}{\partial y} F(X_s, Y_s) dY_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} F(X_s, Y_s) d\langle X\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2} F(X_s, Y_s) d\langle Y\rangle_s + \int_0^t \frac{\partial^2}{\partial x \partial y} F(X_s, Y_s) d\langle X, Y\rangle_s$$

Let us denote:

$$Z_t = \frac{X_t}{1+Y_t}.$$

6. Prove that Z satisfies

$$dZ_t = (\alpha Z_t - Z_t^2)dt + \sigma Z_t dW_t. \tag{1}$$

Proof. We first remark that $d\langle X, Y\rangle_s \equiv 0$ and $d\langle Y\rangle_s \equiv 0$ and $\frac{\partial^2}{\partial x^2}F \equiv 0$. We have

$$\begin{aligned} Z_t &= Z_0 + \int_0^t \frac{1}{1+Y_t}(\alpha X_t dt + \sigma X_t dW_t) + \int_0^t -\frac{X_t}{(1+Y_t)^2}dY_t \\ &= Z_0 + \int_0^t (\alpha Z_t - Z_t^2)dt + \int_0^t \sigma Z_t dW_t. \end{aligned}$$

□

We denote by $p(t,z)$ the density of the solution (1).

7. Give the Fokker-Planck equation satisfied by p.

Proof. The pdf solves

$$\begin{aligned} \frac{\partial}{\partial t}p(t,z) &= L^* p \\ &= -\left((\alpha z - z^2)p(t,z)\right)' + \frac{\sigma^2}{2}\left(z^2 p(t,z)\right)'' \end{aligned}$$

□

We now assume that $\alpha > \frac{\sigma^2}{2}$.

8. Give the invariant measures.

Proof. A simple computation gives

$$\frac{p'}{p} = \frac{\frac{2\alpha}{\sigma^2} - 2}{z} - \frac{2}{\sigma^2}.$$

So, we obtain

$$p_\alpha(z) = \begin{cases} N_\alpha z^{\frac{2\alpha}{\sigma^2}-2}\exp\left(-\frac{2z}{\sigma^2}\right), & z > 0 \\ 0, & z \le 0. \end{cases}$$

where N_alpha is a normalization.

□

3 Feed-forward networks

This part concerns the study of feed-forward chains that are used to model synchronization in neural networks. We consider the system in $\mathbb{R}^m \times \mathbb{R}^m$ given by

$$\begin{aligned} \dot{x}_2 &= f(x_2, 0) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned} \tag{2}$$

where $f(0,0) = 0$. We further assume that the linearisation at the origin is $J = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$.

9. Show that the flow can be written $\Phi^t(x_2, x_3) = (\phi^t(x_2, 0), \phi^t(x_3, x_2))$

Proof. The flow can be written $\Phi^t(x_2, x_3) = (\psi^t(x_2), \phi^t(x_3, x_2))$ with $\frac{d\psi^t}{dt}(x_2)|_{t=0} = f(x_2, 0)$ and $\frac{d\phi^t}{dt}(x_3, x_2)|_{t=0} = f(x_2, x_3)$. Therefore $\frac{d\psi^t}{dt}(x_3)|_{t=0} = \frac{d\phi^t}{dt}(x_3, 0)|_{t=0}$. Since this holds for all x_3, it follows that $\frac{d\psi^t}{dt}(x_3) = \frac{d\phi^t}{dt}(x_3, 0)$. Thus $\psi^t(x_3) = \phi^t(x_3, 0)$. □

10. We assume that the center subspace $\mathbf{E}_c$ of a is the of dimension n. Show that (2)-a has a locally invariant manifold $\mathcal{V}_c$. Show that (2) has a locally invariant manifold. Show that $\mathcal{V}_c \times \{0\}$ (resp. $\{0\} \times \mathcal{V}_c$) is flow invariant.

Proof. Take $\mathcal{V}_c$ as a local center manifold for (2)-a. Idem but this time for (2), we call it $\mathcal{W}_c$. If we consider $\mathcal{V}_c \times \{0\}$, it is invariant for (2). □

11.a. Let us define the projection $\pi_2(x_2, x_3) = (x_2, 0)$. Show that $\pi_2^{-1}(\mathcal{V}_c \times \{0\})$ is flow invariant.

Proof. We have $\pi_2^{-1}(\mathcal{V}_c \times \{0\}) = \mathcal{V}_c \times \mathbb{R}^m$. It follows from the first question that it is flow invariant. □

11.b. Show that one can choose a 2n-dimensional center manifold $\mathcal{W}_c$ for (2) so that $\pi_2(\mathcal{W}_c) = \mathcal{V}_c \times \{0\}$. (One can reduce $\mathcal{V}_c$.)

12.a. We admit that there is a invertible mapping $P : \mathcal{V}_c \times \mathcal{V}_c \to \mathcal{W}_c$ such that $P : (z_2, z_3) = (z_2, \rho(z_2, z_3))$, $P(z_2, 0) = (z_2, 0)$ and $P(0, z_3) = (0, z_3)$. Compute the flow of (2) on $\mathcal{V}_c \times \mathcal{V}_c$ as function of ϕ_t and the inverse of P.

Proof. We write $P^{-1}(z_2, z_3) = (z_2, \sigma(z_2, z_3))$. Let us look at $\Psi^t(z_2, z_3) = P^{-1}\Phi^t P(z_2, z_3)$. One finds

$$\begin{aligned} \Psi^t(z_2, z_3) = P^{-1}\Phi^t(z_2, \rho(z_2, z_3)) = P^{-1}(\phi^t(z_2, 0), \phi^t(\rho(z_2, z_3), z_2) \\ = (\phi^t(z_2, 0), \sigma(\phi^t(z_2, 0), \phi^t(\rho(z_2, z_3), z_2))). \end{aligned} \tag{3}$$

□

12.b. Deduce that the dynamics on the center manifold $\mathcal{W}_c$ of (2) can be written on $\mathcal{V}_c \times \mathcal{V}_c$ as

$$\begin{aligned} \dot{z}_2 &= g(z_2, 0) \\ \dot{z}_3 &= g(z_3, z_2) \end{aligned} \tag{4}$$

for some function g and coordinates $z_2, z_3 \in \mathcal{V}_c$. (One could use question 3.1)

Proof. We write $\psi^t(z_2, z_3) := \sigma(\phi^t(z_2, 0), \phi^t(\rho(z_2, z_3), z_2))$. We find

$$\psi^t(z_3, 0) = \sigma(\phi^t(0, 0), \phi^t(\rho(0, z_3), 0)) = \sigma(0, \phi^t(z_3, 0)) = \phi^t(z_3, 0) \tag{5}$$

which implies, by question 1, that the differential equations can be written as stated. ∎

We want to look at the normal form.

13.a Assume that the center subspace $\mathbf{E}_c$ *is two-dimensional, and the linearization of the internal dynamics a has a pair of purely imaginary eigenvalues. We rescale time so these are equal to* $\pm i$*. Does the normal form has the same structure as* (6)*?*

Proof. Not necessarily. Applying the normal form theorem does not respect the structure of these equations. ∎

13.b Show that any change of variable $(z_2, z_3) \to (Q(z_2), Q(z_3))$ *which leaves* $(0, 0)$ *invariant does change the strcture of* (4)*. Deduce that we can assume* $g(z_2, 0)$ *being in normal form. What is the expression of this normal form?*

Proof. First part is simple. We recognize a Hopf bifurcation. The normal form is like $\dot{z}_2 = z_2 P(|z_2|^2) + h.o.t.$ ∎

13.c We assume that for any value of k*, there exists a polynomial change of coordinates which can transform (4) into the system in* $\mathbb{C} \times \mathbb{C}$

$$\begin{aligned} \dot{z}_2 &= p_k(z_2, \overline{z_2}) + h.o.t. \\ \dot{z}_3 &= p_k(z_3, \overline{z_3}) + q_k(z_3, \overline{z_3}, z_2, \overline{z_2}) + h.o.t. \end{aligned} \tag{6}$$

where p_k *and* q_k *are polynomials of order* k*,* $q_k(z_3, \overline{z_3}, 0, 0) = 0$*, h.o.t. indicates terms of degree at least k + 1 and the truncated system is equivariant under the action of* $\mathbf{S}^1$*,* $\theta \cdot (z_2, z_3) = (e^{i\theta} z_2, e^{i\theta} z_3)$*. What is the general expression for* p_k, q_k*?*

Proof. p_k satisfies the same relation as in the case of the Hopf bifurcation. Hence $p_k(z) = z \cdot \tilde{p}_k(|z|^2)$. We have $q_k(e^{i\theta} z_2, e^{i\theta} z_3) = e^{i\theta} q_k(z_2, z_3)$. Looking at monomials $z_2^a \overline{z_2}^b z_3^c \overline{z_3}^d$ one finds the expression $a - b + c - d = 1$. We write $a - b - 1 = l \in \mathbb{Z}$ and $c - d = -l$. One finds, for $l \geq 0$, the terms $z_2^{1+l} |z_2|^{2b} \overline{z_2}^l |z_3|^{2c} \bar{z}_3^l$. If $l = -1 - \tilde{l} < 0$, we find $a + \tilde{l} = b, c = d + 1 + \tilde{l}$: it gives terms like $\bar{z}_2^{\tilde{l}} |z_2|^{2a} \overline{z_2}^l |z_3|^{2d} z_3^{\tilde{l}+1}$. Hence, we find that $q_k = z_2 A(|z_2|^2, |z_3|^2, z_2 \bar{z}_3) + z_3 B(|z_2|^2, |z_3|^2, \bar{z}_2 z_3)$ where A, B are polynomials. ∎