Paris 6 Master Mathbio / ENS / Master MVA 2016-2017
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Exam - Mathematical Methods in Neuroscience
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Lecture questions

1. Give the Fold and Pitchfork normal forms. Analyse the truncated normal forms (one can assume real coefficients for the normal form).

2. Give the Hopf normal form at order 3. Analyse the truncated normal form (one can assume real coefficients for the normal form).

3. Consider a continuous time Markov Process $(X_t)_{t\geq 0}$ with values in the finite set $\{0,1\}$. Assume the jump rates are constant in time, say $r_{1,0}$ to jump from state 1 to state 0 and $r_{0,1}$ to jump from state 0 to state 1. Give the infinitesimal generator of the Markov process X. Do not forget its domain.

Exercices

Ex. 1

Let $(\varphi(t))_{t\geq 0}$ be a continuous positive function. For all $N \in \mathbb{N}^*$, define the sequence $(\varphi_k^N)_{k\geq 0}$ by

$$\varphi_k^N = N\varphi(\frac{k}{N})$$

Consider the continuous time Markov Process $(X_t^N)_{t\geq 0}$ with values in $\mathbb{N}$ defined by:
X_t^N is piecewise constant. It jumps from state k to state $k+1$ with rate φ_k^N. Let $Y^N(t) = \frac{X_t^N}{N}$.

4. Give the infinitesimal generator of Y^N.

5. Does the process Y^N converge as N goes to infinity? In the case it does, give the dynamic of the limit.

Ex. 2

Let $\varepsilon > 0$ and $(X_t^\varepsilon)_{t\geq 0}$ be the solution of the linear stochastic differential equation

$$X_t^\varepsilon = x_0 + \int_0^t \lambda(A - X_s^\varepsilon)ds + \sqrt{\varepsilon}W_t,$$

where $(W_t)_{t\geq 0}$ is a standard one dimensional Brownian motion, $\lambda > 0$ and A are constant.

6. Give for all $t \in [0,T]$ the law of X_t^ε.

7. Let $0 \le s < t \le T$. Give the law of $(X_s^\varepsilon, X_t^\varepsilon)$.
8. Give the asymptotic behaviour of X_T^ε as T goes to infinity.
9. Give the asymptotic behaviour of $(X_t^\varepsilon)_{0\le t\le T}$ as ε goes to 0.
10. Using Freidlin Wentzell Theorem, evaluate

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\sup_{0\le t\le T} |X_t^\varepsilon - X_t^0| \ge \eta\right),$$

where $\eta > 0$ is fixed.

Problem

We consider the delay differential equation

$$\text{(E)} \qquad \dot{V} = f(V(t), V(t-D)) \in \mathbb{R}.$$

We assume that f is globally Lipschitz on the first variable and continuous on the second. We also assume that the constant delay D is positive $D > 0$. The goal of this part is to provide an "elementary" way of looking at the dynamics of (E).

11. Give examples of delays in the modelling of neurons.
12. Is the problem (E) finite dimensional? Give a plausible state space. Can we take D as a bifurcation parameter?
13. Show that we can assume $D = 1$.

We now assume in the following that $D = 1$.

14. We write $\mathcal{C} := C([0,1])$ and $G : \mathcal{C} \to \mathcal{C}$ is the mapping $G(\phi) = v$ where v solves

$$\text{(1)} \qquad \begin{cases} \dot{v}(t) = F(v(t), \phi(t)), & t \in [0,1] \\ v(0) = \phi(1). \end{cases}$$

Show that there is a unique solution to (1).

15. Express the solution of (E) with G.

We assume in the following that F is C^1 on $\mathbb{R}^2$, we also assume that G is C^1.

16. If (E) has an equilibrium v^f, what does it implies for G? What is a sufficient condition for v^f to be stable?
17. Show that the differential $u = dG(\phi) \cdot \delta \in \mathcal{C}$ of G evaluated at $\phi \in \mathcal{C}$ and applied to $\delta \in \mathcal{C}$ solves $\dot{u} = a(t)u(t) + b(t)\delta(t)$ with $u(0) = \delta(1)$ for some a, b to be precised.

We want to compute the spectrum of $dG(v^f) \cdot \delta$ to get a condition for having a stable equilibrium.

18. Express the eigenvalues μ of $dG(v^f) \cdot \delta$ as function of the solutions to the equation $\lambda = a + be^{-\lambda}$.
19. Compute the spectrum of $dG(v^f) \cdot \delta$, show that it is only composed of eigenvalues.
20. Find a condition on a, b for a Hopf bifurcation to occur for (E). What does it mean for G? Can you give the normal form?