EXAMEN MVA 2013

In all the document, we work in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ assumed to satisfy the usual conditions. We are interested in the large scale behavior arising from the nonlinear coupling of a large number N of stochastic diffusion processes representing the membrane potential of neurons in the framework of rate models. Hence the variable characterizing the neuron state is its firing rate, that exponentially relaxes to zero when it receives no input, and that integrates both external input and the current generated by its neighbors.

1. First properties of the network equations. The network is composed of P neural populations that differ by their intrinsic dynamics, the input they receive and the way they interact with the other neurons. Each population $\alpha \in \{1, \cdots, P\}$ is composed of N_α neurons, and we assume that the ratio N_α/N converges to a constant $\delta_\alpha \in]0, 1[$ when the total number of neurons N becomes arbitrarily large. We define the population function p that maps the index $i \in 1, \cdots, N$ of any neuron to the index α of the population neuron i belongs to: $p(i) = \alpha$.

These sigmoidal functions $S_\beta : \mathbb{R} \to \mathbb{R}$ are assumed to be smooth (Lipschitz continuous), increasing functions that tend to 0 at $-\infty$ and to 1 at ∞. The firing rate of the presynaptic neuron j, multiplied by the synaptic weight J_{ij}, is an input current to the postsynaptic neuron i. We assume that the synaptic weight J_{ij} is equal to $J_{p(i)p(j)}/N_{p(j)}$.

The network behavior is therefore governed by the following set of stochastic differential equations:

$$d\bar{V}_i(t) = \left[-\frac{1}{\tau_\alpha}\bar{V}^i(t) + I_\alpha(t) + \sum_{\beta=1}^{P} \frac{J_{\alpha\beta}}{N_\beta} \sum_{j:p(j)=\beta} S_\beta(V^j(t)) \right] dt + \lambda_\alpha(t) dB^i(t) \tag{1.1}$$

which are the sum of a **deterministic** part $I_\alpha(t)$ and a stochastic additive noise driven by N **independent adapted Brownian** motions $(B^i)_{i=1\ldots N}$ modulated by the **deterministic** diffusion coefficients $\lambda_\alpha(t)$.

1. Show that for all $T > 0$, (1.1) has a unique solution defined for $t \in [0, T]$ when $V_i(0) \in \mathrm{L}^2(\Omega, \mathbb{P})$.

Proof. To apply the existence/uniqueness theorem, we need to show that the drift and the noise coefficient are globally Lipschitz continuous and linearly bounded in V^i. This is obvious for the noise term. Thanks to the properties of the sigmoid function, this is also the case for the drift term. □

2. Gronvall.??

We introduce another variable $\bar{V}_\alpha$ solution of the stochastic differential equation:

$$d\bar{V}_\alpha(t) = \left[-\frac{1}{\tau_\alpha}\bar{V}_\alpha(t) + I_\alpha(t) + \sum_{\beta=1}^{P} J_{\alpha\beta}\mathbb{E}\left[S_\beta(\bar{V}_\beta)\right] \right] dt + \lambda_\alpha(t) dB^\alpha(t)$$
$$= b_\alpha(\bar{V})dt + \lambda_\alpha(t)dB^\alpha(t) \tag{1.2}$$

Recall that by assumptions the B^α are independent as a subfamily of (B^i).

3. Assume that (1.2) has a unique solution in the class of Ito processes. Show that, if the initial condition is $\bar{V}_1(0)$ and d composed of independent random variables, $\bar{V}_\alpha(0)$, then

Proof. □

2. Analysis of the McKean equations. The goal of the next ?? questions is to prove that (1.2) has a unique solution. To simplify, we assume that there is a unique population, hence $P = 1$. Let us introduce complete vectorial space

$$\mathcal{E} = \left\{ (X_s)_{0\le s\le T}, \text{ continuous process, } \mathcal{F}_t \text{ adpated such that } \mathbb{E}\left(\sup_{s\le T}|X_s|^2\right) < \infty \right\}$$

with the norm $\|X\| = \sqrt{\mathbb{E}\left(\sup_{0\le s\le T}|X_s|^2\right)}$

3. Show that there a constant K such that $\|b(X) - b(Y)\| \le K\|X - Y\|$ and $\|b(X)\| \le K(1 + \|X\|)$

Proof. We have:

$$\begin{aligned}|b(X_t) - b(Y_t)|^2 =& |\mathbb{E}\left(S(X_t) - S(Y_t)\right)|^2 \le K_S^2|\mathbb{E}\left(X_t - Y_t\right)|^2 \\ &\le K_S^2\mathbb{E}\left(X_t - Y_t\right)^2 \le K_S^2\mathbb{E}\sup_{t\le T}\left(X_t - Y_t\right)^2 = K_S^2\|X - Y\|^2.\end{aligned}$$

Also, we find

$$\begin{aligned}|b(X_t)|^2 =& |\mathbb{E}\left(S(X_t)\right)|^2 \le K_S^2|\mathbb{E}\left(1 + |X_t|\right)|^2 \overset{(a+b)^2\le 2(a^2+b^2)}{\le} 2K_S^2\mathbb{E}\left(1 + X_t^2\right) \\ &\le 2K_S^2\mathbb{E}\left(1 + \sup_t X_t^2\right) = 2K_S^2\left(1 + \|X\|^2\right).\end{aligned}$$

This allows to conclude. ∎

3. Show that $\Phi(X)_t = Z + \int_0^t b(X_s)ds + \int_0^t \lambda_\alpha(s)dB^\alpha(s)$ is Lipschitz with constant $kT^{3/2}$.

Proof. We have

$$|\Phi(X)_t - \Phi(Y)_t|^2 \le \sup_{t\le T}\left(\int_0^t b(X_s) - b(Y_s)ds\right)^2 \le \sup_{t\le T}K^2\left(\int_0^t \mathbb{E}\left(X_s - Y_s\right)ds\right)^2.$$

From Cauchy-Schwartz, we have

$$|\Phi(X)_t - \Phi(Y)_t|^2 \le \sup_{t\le T}K^2T^2\int_0^t\left(\mathbb{E}\left(X_s - Y_s\right)\right)^2ds \le \sup_{t\le T}K^2T^2\int_0^t \mathbb{E}\left(X_s - Y_s\right)^2ds.$$

Also

$$\int_0^t \mathbb{E}\left(X_s - Y_s\right)^2ds \le \int_0^t \mathbb{E}\sup_{s\le T}|X_s - Y_s|^2ds \le T\mathbb{E}\sup_{s\le T}|X_s - Y_s|^2.$$

which gives

$$\mathbb{E}\sup_{t\ge T}|\Phi(X)_t - \Phi(Y)_t|^2 \le K^2T^3\|X - Y\|^2.$$

∎

3. Show that $\|\Phi(0)\|^2 < \infty$. Conclude that Φ maps $\mathcal{E}$ to $\mathcal{E}$.

Proof. Using $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, we have

$$|\phi(0)_t|^2 \le 3\left(Z^2 + \sup_{t\le T}|\int_0^t b(0)ds|^2 + \sup_{t\le T}|\int_0^t \lambda(s)dB(s)|^2\right)$$

Using $\left\|\int_0^t \lambda(s)dB(s)\right\| \leq 4\mathbb{E}\left(\int_0^T \lambda(s)^2 ds\right) \leq 4K_B^2 T$, we find the requested inequality.

The process $\int_0^t \lambda_\alpha(s)dB^\alpha(s)$ is continuous and adapted by the Ito integral construction. The process $\int_0^t b(X_s)ds$ is continuous because X_s is continuous and is adapted because X_s is adapted.

Hence, Φ maps $\mathcal{E}$ to $\mathcal{E}$ because it maps every balls centered on 0 into balls centered on $\Phi(0) \in \mathcal{E}$. ☐

3. Prove that (1.2) has a unique solution in $\mathcal{E}$

Proof. We note $k(T) = KT$ the Lipschitz constant of Φ which we assume $k(T) < 1$. Then Φ is contracting from $\mathcal{E}$ to $\mathcal{E}$: it has a unique fixed point in $\mathcal{E}$. This proves the existence of a solution to (1.2). To prove it for a general T, it is enough to reiterate the argument on time intervals of length $1/2K$. ☐

3. Consider a solution X_t of (1.2). We introduce $T_n = \inf\{s \geq 0,\ |X_s| > n\}$ and $f^n(t) = \mathbb{E}\sup_{0\leq s\leq \min(t,T_n)} |X_s|^2$. Show that $f^n(t) \leq a + b\int_0^t f^n$ for some positive a, b, functions of T, but independent of n. Conclude that $f^n(T) < K_2$ where K_2 is function of T but is independent of n. Conclude that (1.2) has a unique solution in the class of Ito processes.

Proof. Using similar bounds as above, we find:

$$f^n(t) \leq 3\left(\mathbb{E}(Z^2) + TK\mathbb{E}\int_0^{\min(t,T_n)} 1 + \mathbb{E}X_s^2 ds + 4K_B^2 T\right) \leq$$

$$3\left(\mathbb{E}(Z^2) + 4K_B^2 T + TK\int_0^t 1 + \mathbb{E}\sup_{s\leq\min(t,T_n)} X_s^2 ds\right)$$

which allows to conclude the first part. Gronwall lemma gives that $f^n(T) \leq a\left(1 + e^{bT}\right)$ which shows that $f^n(T) < K_2 < \infty$. Using Fatou lemma, we take the limit $n \to \infty$ for T fixed and find

$$\mathbb{E}\left(\sup_{s\leq T} |X_s|^2\right) < K_2 < \infty$$

which shows that $X_s \in \mathcal{E}$. ☐

3. A look at the propagation of chaos]. We come back to the origin problem now.

3. Show that $\mathbb{E}\left\{\max_\alpha\left(\frac{1}{N}\int_0^t e^{-(t-s)/\tau_\alpha}\sum_{j=1}^N \left(S_{p(j)}(\bar{V}_s^j) - \mathbb{E}[S_{p(j)}(\bar{V}_s^j)]\right) ds\right)^2\right\} \leq \frac{\tau_\alpha^2}{N}$

Proof. Since the variables $S_{p(j)}(\bar{V}_s^j) - \mathbb{E}[S_{p(j)}(\bar{V}_s^j)]$ are independent and centered and

$S_\alpha \in (0,1)$, we have

$$\mathbb{E}\left\{\max_\alpha\left(\frac{1}{N}\int_0^t e^{-(t-s)/\tau_\alpha}\sum_{j=1}^N\left(S_{p(j)}(\bar V_s^j)-\mathbb{E}[S_{p(j)}(\bar V_s^j)]\right)ds\right)^2\right\}$$
$$=\frac{1}{N^2}\sum_{j=1}^N\mathbb{E}\left[\max_\alpha\left(\int_0^t e^{-(t-s)/\tau_\alpha}\left(S_{p(j)}(\bar V_s^j)-\mathbb{E}[S_{p(j)}(\bar V_s^j)]\right)ds\right)^2\right]$$
$$\leq \frac{1}{N^2}\sum_{j=1}^N\left(\int_0^t e^{-(t-s)/\tau_\alpha}\right)\leq\frac{\tau_\alpha^2}{N}$$

□

3. Write $\tau = \max_\alpha \tau_\alpha$ and define $M_t = \mathbb{E}\left(\max_i |V_t^i - \bar V_t^i\right)$. Show that $M_t \leq K\int_0^t e^{-(t-s)/\tau}M_s ds + K'\frac{\tau}{\sqrt{N}}$ for some constants K, K'.

Proof. From:

$$V_t^i - \bar V_t^i = \sum_{\beta=1}^P J_{\alpha\beta}\int_0^t e^{-(t-s)/\tau_\alpha}\frac{1}{N_\beta}\sum_{j:p(j)=\beta}\left\{\left(S_\beta(V_s^j)-S_\beta(\bar V_s^j)\right)+\left(S_\beta(\bar V_s^j)-\mathbb{E}S_\beta(\bar V_s^j)\right)\right\}ds$$

we find:

$$|V_t^i-\bar V_t^i| \leq K\int_0^t e^{-(t-s)/\tau}\max_{j=1\ldots N}|V_s^j-\bar V_j^s| + K'\left|\frac{1}{N}\int_0^t e^{-(t-s)/\tau_\alpha}\sum_{j=1}^N\left(S_\beta(\bar V_s^j)-\mathbb{E}S_\beta(\bar V_s^j)\right)ds\right| \tag{3.1}$$

where $K = \max_\alpha \sum_\beta |J_{\alpha\beta}|L$ and L is the largest Lipschitz constant of the sigmoids and $K' = \max_\alpha\sum_\beta |J_{\alpha\beta}|\max_\beta N/N_\beta$ quantity upperbounded, for N sufficiently large, by $\max_\alpha\sum_\beta|J_{\alpha\beta}|\max_\beta 2/\delta_\beta$. Since the righthand side of the previous equation does not depend on the index i, taking the maximum with respect to i and the expected value of both sides, we obtain:

$$M_t \leq K\int_0^t e^{-(t-s)/\tau}\mathbb{E}\max_{j=1\ldots N}|V_s^j-\bar V_j^s| + K'\mathbb{E}\max_\alpha\left|\frac{1}{N}\int_0^t e^{-(t-s)/\tau_\alpha}\sum_{j=1}^N\left(S_\beta(\bar V_s^j)-\mathbb{E}S_\beta(\bar V_s^j)\right)ds\right|$$

Now, using the previous question, we get

$$M_t \leq K\int_0^t e^{-(t-s)/\tau}M_s ds + K'\frac{\tau}{\sqrt{N}}$$

□

3. Conclude that the almost sure convergence of V_t^i (as a random variable) to $\bar V_t^i$ is uniform in time, i.e. $\sup_{0\leq t\leq T}\mathbb{E}\left(|V_t^i - \bar V_t^i|\right) \leq \frac{C'}{\sqrt{N}}$

Proof. This is Gronwall. □

3. Show that $\mathbb{E}\left(\sup_{0\leq t\leq T}|V_t^i-\bar V_t^i|\right)\leq\frac{C(T)}{\sqrt{N}}$

Proof. Upperbounding the exponential term in (3.1) by 1 and taking the supremum, it is easy to see that

$$\mathbb{E}\left[\sup_{t\leq T}\max_i |V_t^i - \bar{V}_t^i|\right] \leq K\int_0^T \mathbb{E}\left[\sup_{s\leq t}\max_j |V_s^j - \bar{V}_s^j|\right] dt + \frac{K'T}{\sqrt{N}}.$$

Using:

$$\mathbb{E}\left[\max_\alpha \sup_{t\in[0,T]}\left(\frac{1}{N}\int_0^t e^{-(t-s)/\tau_\alpha}\sum_{j=1}^N \left(S_{p(j)}(\bar{V}_s^j) - \mathbb{E}[S_{p(j)}(\bar{V}_s^j)]\right) ds\right)^2\right]$$
$$\leq \frac{T}{N^2}\int_0^T \mathbb{E}\left[\left|\sum_{j=1}^N \left(S_{p(j)}(\bar{V}_s^j) - \mathbb{E}[S_{p(j)}(\bar{V}_s^j)]\right)\right|^2\right] ds$$
$$= \frac{T}{N^2}\sum_{j=1}^N\int_0^T \mathbb{E}\left[\left|\left(S_{p(j)}(\bar{V}_s^j) - \mathbb{E}[S_{p(j)}(\bar{V}_s^j)]\right)\right|^2\right] ds \leq \frac{T}{N^2}$$

using the independence of the $\bar{V}^j$ and Cauchy-Schwartz inequality. This last estimate readily implies, using Gronwall's inequality:

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\max_i |V_t^i - \bar{V}_t^i|\right] \leq \frac{K'Te^{KT}}{\sqrt{N}}$$

□

3. Conclude that $(V_t^i)_{t\in[0,T]}$ converges a.s. towards the process $(\bar{V}_t^\alpha)_{t\in[0,T]}$.

Proof. The last question is the definition of such convergence. □

3. Show that provided that the initial conditions of all neurons are independent, the law of $(V^{i_1}(t),\dots,V^{i_k}(t), t\leq T)$ for any fixed $k\geq 2$ and $(i_1,\dots,i_k)\in\mathbb{N}^k$, converges towards $\nu_{p(i_1)}\otimes\cdots\otimes\nu_{p(i_k)}$ when $N\to\infty$ where ν_α is the law of the solution of equation (1.2) corresponding to population α. This means that $(V^{i_1}(t),\dots,V^{i_k}(t))$ become independent processes.

Proof. From the previous question, we have the almost sure convergence of $(V^{i_1}(t),\dots,V^{i_k}(t), t\leq T)$ towards $(\bar{V}^{i_1}(t),\dots,\bar{V}^{i_k}(t), t\leq T)$, this implies the convergence in law. Now, if the initial conditions are independent, the processes $\bar{V}^{i_k}$ □

4. Analysis of mean field equations. We assume, in the rest of the document, that $\bar{V}^0 = (\bar{V}_\alpha^0)_{\alpha=1\cdots P}$ is a P-dimensional Gaussian random variable.

3. Show that the solutions of the P mean field equations (1.2) with initial conditions $\bar{V}^0$ are Gaussian processes for all time.

Proof.

□

3. Let $\mu_\alpha(t)$ be the mean of the process $\bar{V}_\alpha(t)$ and $v_\alpha(t)$ its variance. Let also $f_\beta(x,y)$ denote the expectation of $S_\beta(U)$ for U a Gaussian random variable of mean x and variance y. Show that

$$\begin{aligned}\dot{\mu}_\alpha(t) &= -\tfrac{1}{\tau_\alpha}\mu_\alpha(t) + \sum_{\beta=1}^P J_{\alpha\beta} f_\beta(\mu_\beta(t), v_\beta(t)) + I_\alpha(t)\\ \dot{v}_\alpha(t) &= -\tfrac{2}{\tau_\alpha}v_\alpha(t) + \lambda_\alpha^2(t)\end{aligned} \tag{4.1}$$

Proof.

□

3. Find the expression of $v_\alpha(t)$*. Show, that if* λ_α *is constant in time,* v_α *converge exponentially fast. Write the equation for* μ_α *assume that* v_α *has converged.*